

LOWEST DEGREE INVARIANT 2ND ORDER PDES OVER RATIONAL HOMOGENEOUS CONTACT MANIFOLDS

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ABSTRACT. For each simple Lie algebra \mathfrak{g} (excluding, for trivial reasons, type C) we find the lowest possible degree of an invariant 2nd order PDE over the adjoint variety in $\mathbb{P}\mathfrak{g}$, a homogeneous contact manifold. Here a PDE $F(x^i, u, u_i, u_{ij}) = 0$ has degree $\leq d$ if F is a polynomial of degree $\leq d$ in the minors of (u_{ij}) , with coefficients functions of the contact coordinates x^i, u, u_i (e.g., Monge–Ampère equations have degree 1). For \mathfrak{g} of type A or G_2 we show that this gives all invariant 2nd order PDEs. For \mathfrak{g} of type B and D we provide an explicit formula for the lowest-degree invariant 2nd order PDEs. For \mathfrak{g} of type E and F_4 we prove uniqueness of the lowest-degree invariant 2nd order PDE; we also conjecture that uniqueness holds in type D.

1. INTRODUCTION

1.1. Starting point. The problem of classifying scalar second order PDEs admitting a large group of symmetries is the basis of a very extensive research programme, originating in the work of Lie, Darboux, Cartan and others. It is naturally broken into the sub-problems of classifying G -invariant PDEs with a prescribed Lie group G of symmetries. This is still a vast project in general. Using the interpretation of 2nd order PDEs in terms of contact structures, we restrict our attention to the case where G acts transitively on the underlying contact manifold M . Even there we are facing a hopelessly complex task, in particular implying the classification of homogeneous contact G -manifolds for the given Lie group. The latter problem is considered by one of the authors in [2]. In the present paper we shall still significantly narrow the focus, by working in the complex holomorphic setting, assuming G to be simple, and requiring that M be compact. There we can use the structure theory of simple complex Lie algebras, and finally reduce the question to an algebraic problem in invariant theory. Since results about the real case can be then recovered (in a straightforward, if laborious, manner) considering suitable real forms of G , from now on we shall apply the term ‘partial differential equation’ to what is more properly its complexification.

1.2. Context. Remarkably, it turns out that in this context—with the exception of groups G of type A and, for trivial reasons, C—a G -invariant second order PDE has precisely the Lie algebra \mathfrak{g} of G as its *local* infinitesimal symmetries at any point of M . This has been observed and used by D. The in [23] to realise the simple Lie algebras not of type C as infinitesimal symmetries of 2nd order PDEs, a problem with a long tradition, inaugurated by the 1893 works by Cartan and Engel,¹ and recently recast in full generality by P. Nurowski in the context of the so-called *parabolic contact geometries* (see [8], sec. 4.2). In this spirit, 2nd order PDEs can be thought of as additional structures on contact manifolds.

1.3. The degree of a 2nd order PDE on a contact manifold. A 2nd order (scalar, in n independent variables and one unknown function) PDE on a $(2n+1)$ -dimensional contact manifold (M, \mathcal{C}) , \mathcal{C} being the contact distribution on M , is, roughly speaking, a first-order condition imposed on the Lagrangian submanifolds of M . Note that the latter are the integral n -dimensional submanifolds of the exterior differential system $(M, \mathcal{I}_{\mathcal{C}})$, where $\mathcal{I}_{\mathcal{C}}$ is the ideal of differential forms vanishing on \mathcal{C} (see [6]). Recall that the Levi (twisted) two-form $\mathcal{C} \wedge \mathcal{C} \rightarrow TM/\mathcal{C}$ is nondegenerate, and as such it defines a conformal symplectic structure on each contact plane. Imposing the aforementioned first-order condition is the same as restricting the prolonged exterior differential system $(M^{(1)}, \mathcal{I}_{\mathcal{C}}^{(1)})$ to a hypersurface \mathcal{E} of the manifold $M^{(1)}$ of the n -dimensional integral elements of $(M, \mathcal{I}_{\mathcal{C}})$. The manifold $M^{(1)}$ has a natural smooth bundle structure

$$(1) \quad \pi : M^{(1)} = \bigcup_{m \in M} \text{LGr}(\mathcal{C}_m) \rightarrow M,$$

such that the fibre of π at $m \in M$ naturally identifies with the *Grassmannian of Lagrangian planes* $\text{LGr}(\mathcal{C}_m)$ of \mathcal{C}_m .

In contact (or Darboux) coordinates (x^i, u, u_i) , a generic Lagrangian submanifold of M is the graph $\Gamma_f^{(1)} := \{x^i, u = f(x^1, \dots, x^n), u_i = f_{x^i}(x^1, \dots, x^n)\}$ of the 1st jet of a function in the n variables x^i . Hence, a first-order condition on Lagrangian submanifolds is a relation between the first derivatives of both f and all the f_{x^i} ’s, that is, a second-order PDE on f . Globally, this corresponds to a hypersurface $\mathcal{E} \subset M^{(1)}$.

One can locally extend the Darboux coordinates to $M^{(1)}$ as follows. The Lagrangian space $L = T_m \Gamma_f^{(1)} \in M^{(1)}$ has coordinates (x^i, u, u_i, u_{ij}) where (x^i, u, u_i) are the coordinates of m , and $T_m \Gamma_f^{(1)} = \langle D_{x^i}|_m + u_{ij} \partial_{u_i}|_m \mid i = 1, 2, \dots, n \rangle$, with

$$(2) \quad D_{x^i} = \partial_{x^i} + u_i \partial_u.$$

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¹See [23] for a more detailed historical account, as well as a broader list of references.

Observe that the u_{ij} -coordinates of $T_m \Gamma_f^{(1)}$ are precisely the second-order derivatives of f at (x^1, \dots, x^n) and, as such, they are symmetric in the indices (i, j) : this corresponds to the canonical identification $T_L \text{LGr}(\mathcal{C}_m) \simeq S^2 L^*$, valid for all $L \in \text{LGr}(\mathcal{C}_m)$. Accordingly, in these coordinates, a 2nd order PDE reads as $\mathcal{E} = \{F(x^i, u, u_i, u_{ij}) = 0\}$.

The intrinsic geometry of $\text{LGr}(\mathcal{C}_m)$ allows us to introduce a point-wise numerical invariant characterising the hypersurface \mathcal{E} : the *degree*. Namely, we say that it is of degree d at $m = (x^i, u, u_i)$ if $F(x^i, u, u_i, u_{ij})$ is a polynomial of degree d as a function of the minors of the symmetric matrix $U = (u_{ij})$. If the number d does not depend on m , we say that the PDE \mathcal{E} has *degree* d , though its *order* is always 2. The chief example when the notion of degree is well-defined, is when there is a group of contactomorphisms acting transitively on \mathcal{E} .

1.4. A first look at the results. Our main results concern the case where (M, \mathcal{C}) is a homogeneous contact manifold for a *complex* simple Lie group G . In fact, requiring M to be compact, it is uniquely determined by G . We consider hypersurfaces $\mathcal{E} \subset M^{(1)}$ that are invariant under the natural lift of the G -action. For some G we give a complete and explicit description of the set of all such hypersurfaces. In general, for each G we characterise the minimal possible *degree* of such a hypersurface.

Since G acts transitively on M by contact transformations, and $\mathcal{E} \subset M^{(1)}$ is supposed to be G -invariant, the problem of describing such hypersurfaces is easily reduced to the study of hypersurfaces in a single fibre $\text{LGr}(\mathcal{C}_m)$ invariant under the subgroup of G stabilising m . Thus our results are in fact about hypersurfaces in a Lagrangian Grassmannian of a (conformal) symplectic vector space, invariant under certain subgroups of the (conformal) symplectic group.

1.5. Plan of the paper. Section 2 gives a more in-depth description of our main results, introducing for the first time all the constructions necessary for a precise statement of Theorem 1. Section 3 provides a careful technical exposition of the underlying material, where we reintroduce and prove many of the standard results mentioned in Section 2. This way we prepare the ground for an algebraic reformulation of the main result, opening Section 4. We then outline the strategy of the proof, based on the classification of complex simple Lie algebras. Parts of the main result corresponding to the different Cartan types A, \dots, G occupy the subsequent subsections. Section 5 contains a discussion of further consequences of our results and their relation to existing research. Finally, in Section 6 we write down some of our invariant PDEs in explicit form.

2. DESCRIPTION OF MAIN RESULTS AND METHODS

2.1. Basic constructions and results. We will now quickly introduce a number of notions necessary for a precise statement of our results. Complete definitions and proofs will be given in Section 3. We state our main Theorem at the end of this subsection.

Definition 1. Let \mathfrak{g} be a complex simple Lie algebra, and G the identity component of $\text{Aut } \mathfrak{g}$. Then the unique closed G -orbit X in $\mathbb{P}\mathfrak{g}$ is called the *adjoint variety* of \mathfrak{g} .

Let \mathfrak{g} be a complex simple Lie algebra other than $\mathfrak{sl}(2, \mathbb{C})$. We let G and $X \subset \mathbb{P}\mathfrak{g}$ be as in Definition 1. We shall fix an origin $o \in X$ and let $P \subset G$ be its stabiliser. The latter is a parabolic subgroup with unipotent radical $P_+ \subset P$ and connected, reductive quotient $G_0 \simeq P/P_+$. We split the projection $P \rightarrow G_0$ by fixing a Levi decomposition

$$(3) \quad P = G_0 \ltimes P_+,$$

thus inducing the so-called *contact grading* (see [8], sec. 3.2.4)

$$(4) \quad \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

of the Lie algebra \mathfrak{g} , where \mathfrak{g}_0 is the Lie algebra of G_0 , and $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ the Lie algebra of P_+ . Considering the graded nilpotent subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \simeq T_o X$ one finds that $\dim \mathfrak{g}_{-2} = 1$ and

$$(5) \quad \dim \mathfrak{g}_{-1} = 2n, \quad n \in \mathbb{N},$$

with the Lie bracket inducing a non-degenerate twisted (i.e., \mathfrak{g}_{-2} -valued) symplectic form on \mathfrak{g}_{-1} .

The hyperplane $\mathfrak{g}_{-1} \subset T_o X$ equips the homogeneous space $X \simeq G/P$ with an invariant contact structure. We thus have a contact manifold as in Subsection 1.3 and we can form the bundle

$$(6) \quad X^{(1)} \longrightarrow X$$

of Lagrangian Grassmannians as in (1). We are interested in (local analytic) hypersurfaces in $X^{(1)}$, locally interpreted as 2nd order scalar PDEs in n independent variables (see again Subsection 1.3). In particular, considering the natural lift of the G -action to $X^{(1)}$, it is natural to ask about the existence and classification of G -invariant hypersurfaces, as these correspond to PDEs with a large symmetry group. More precisely, we consider the family

$$\text{Inv}(X, G) = \left\{ \mathcal{E} \mid \mathcal{E} \text{ hypersurface in } X^{(1)} \text{ such that } G \cdot \mathcal{E} = \mathcal{E} \right\},$$

where from now on a *hypersurface* is *closed*, hence algebraic. In fact, we shall abuse the terminology by allowing the components of our hypersurfaces to have non-negative multiplicities, whence the proper term would be an *effective divisor*. Doing so, we stick to the usual language of geometric PDE theory without introducing unnecessary restrictions on the algebraic side. This abuse does not affect our results.

By virtue of transitivity, the elements of $\text{Inv}(X, G)$ can be put in a natural one-to-one correspondence with the P -invariant hypersurfaces in the fibre $X_o^{(1)}$. Observe also that $X_o^{(1)}$ is the *Lagrangian Grassmannian* of the (twisted)

symplectic space \mathfrak{g}_{-1} , naturally embedded in the *projectivised Plücker space* $\mathbb{P}\Lambda_0^n \mathfrak{g}_{-1}$, where $\Lambda_0^n \mathfrak{g}_{-1}$ is the kernel of the map

$$(7) \quad \Lambda^n \mathfrak{g}_{-1} \rightarrow \Lambda^{n-2} \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}$$

induced by the (twisted) symplectic form. This embedding corresponds to the line bundle $\mathcal{O}(1)$ on $X_o^{(1)}$, generating its Picard group. We then meet again the same notion of *degree*, introduced earlier in Subsection 1.3: a hypersurface in $X_o^{(1)}$ has degree d if it is cut out by a global section of $\mathcal{O}(d)$ or, equivalently, is an intersection of Plücker-embedded $X_o^{(1)}$ with a degree d hypersurface in $\mathbb{P}\Lambda_0^n \mathfrak{g}_{-1}$ (see Definition 5 later on). Given a hypersurface $\mathcal{E} \in \text{Inv}(X, G)$, we refer to the degree of $\mathcal{E}_o \subset X_o^{(1)}$ as the degree of \mathcal{E} itself. This is compatible with the definition given in Subsection 1.3 in terms of minors of the Hessian.

There is a natural geometric construction, appearing for the first time in [23], producing an invariant hypersurface for every \mathfrak{g} of type not C. We review it in Section 5 under the name of the *Lagrangian Chow transform* of the *subadjoint variety*. Knowing thus that $\text{Inv}(X, G)$ is nonempty, and that its elements are grouped by degree, we consider the following natural questions: first, *find the minimum degree* of an element; then, whenever it is possible, *establish whether or not there is a unique element of this degree*. We may now state our main result.

Theorem 1. *The minimal degree of an element of $\text{Inv}(X, G)$ is given by the first row of the following table, while the second row gives the number of elements of that degree (entries marked with an asterisk are conjectural).*

type	A.	B.	D.	E ₆	E ₇	E ₈	F ₄	G ₂
	1	4	2	2	2	2	4	3
	2	<i>unknown</i>	1*	1	1	1	1	1
	2	$2(n-1)$		42	286	13188	16	3

(For comparison, the last row gives the degree of the Lagrangian Chow transform of the subadjoint variety, where n is defined by (5), see Section 5).

Remark 1. Let us explain the absence of type C from the classification. Since in that case $G = \text{Sp}_{n+2}$ acts transitively on $X^{(1)}$, there are no G -invariant hypersurfaces whatsoever, i.e., $\text{Inv}(X, G) = \emptyset$.

2.2. Further constructions and additional results. Our approach relies on translating the geometric problems associated with $\text{Inv}(X, G)$ to algebraic problems of the theory of invariants of the semisimple part of G_0 . We will now give an informal description of this transition. The subtler parts of it, namely equations (9) and (13), will be stated and proved as standalone results in Section 3 below (see Lemma 10 (3) and Proposition 1). As they become rather technical, here we mostly wish to motivate their use.

We have already reformulated a problem involving G -invariance in terms of P -invariance. But since P_+ acts trivially on \mathfrak{g}_{-1} , we may further replace the P -invariance condition with G_0 -invariance or, more accurately, G_0 -equivariance. Note that the linear action of G_0 on $\Lambda_0^n \mathfrak{g}_{-1}$ induces an action on $\mathcal{O}(1)$ over $X_o^{(1)}$. Elements of $\text{Inv}(X, G)$ of degree d are thus in one-to-one correspondence with the nonzero G_0 -equivariant global sections of the line bundles $\mathcal{O}(d)$, $d > 0$, modulo the action of the multiplicative group \mathbb{C}^\times . An element $f \in \Gamma(X_o^{(1)}, \mathcal{O}(d))$ is called G_0 -equivariant if there exists a group homomorphism $\xi : G_0 \rightarrow \mathbb{C}^\times$ satisfying

$$(8) \quad g^* f = \xi(g) f, \quad \forall g \in G_0.$$

Thanks to this re-interpretation, our problem can be formulated in purely algebraic terms. First, we identify the spaces of global sections of $\mathcal{O}(d)$ with the homogeneous summands of the homogeneous coordinate ring of $X_o^{(1)}$, viz.

$$(9) \quad \bigoplus_{d \geq 0} \Gamma(X_o^{(1)}, \mathcal{O}(d)) \simeq (S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^*) / I,$$

where S^\bullet denotes the symmetric algebra, and I is the homogeneous ideal of $X_o^{(1)}$ in its Plücker embedding (this holds by virtue of projective normality, see Sec. 3). In principle, it remains to take the submodule of G_0 -equivariant elements in the right-hand side of (9), decompose it into irreducible submodules, and then detect the one-dimensional ones. But this has to be done carefully, by spelling out the notion of G_0 -equivariance, and eventually passing to the more comfortable context of G_0^{ss} -modules. Here G_0^{ss} is the semisimple part in the decomposition

$$(10) \quad G_0 = G_0^{\text{ss}} \times T$$

of G_0 . The other factor is the central torus $T \simeq (\mathbb{C}^\times)^r$, i.e., the identity component of the centre of G_0 . The integer $r = \text{rk } T$ is the *rank* of T .

Observe that the homomorphism ξ involved in the definition (8) of G_0 -equivariance acts trivially on G_0^{ss} and can thus be regarded as an element of the lattice

$$\widehat{T} := \text{Hom}(T, \mathbb{C}^\times) \simeq \mathbb{Z}^r, \quad r = \text{rk } T$$

of *characters* of T . Since the sub-ring

$$(11) \quad R := (S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^* / I)^{G_0^{\text{ss}}}$$

made of the G_0^{ss} -invariants (i.e., $\mathfrak{g}_0^{\text{ss}}$ -invariants, since G_0^{ss} is connected) is also T -invariant in $S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^*/I$, we can decompose it into irreducible T -submodules:

$$(12) \quad R = \bigoplus_{\xi \in \widehat{T}} R_\xi.$$

Remark 2. If $r = 1$ (resp., $r = 2$), then we can identify \widehat{T} with \mathbb{Z} (resp., with $\mathbb{Z} \times \mathbb{Z}$), in such a way that $R_\xi \subset \Gamma(X_o^{(1)}, \mathcal{O}(\frac{1}{n}\xi))$ (resp., $R_{(\xi_1, \xi_2)} \subset \Gamma(X_o^{(1)}, \mathcal{O}(\frac{1}{n}\xi_1))$). As we shall see below, the case $r = 2$ occurs only in type A, otherwise $r = 1$. That is, we can regard the multi-grading (12) as the usual grading, except in type A, when it is a bi-grading refining the usual grading.

We are finally in position to recast the geometric problem of studying G -invariant hypersurfaces of $X_o^{(1)}$ in the algebraic terms of the (bi)graded ring R of $\mathfrak{g}_0^{\text{ss}}$ -invariants:

$$(13) \quad \text{Inv}(X, G) \simeq \coprod_{\xi \in \widehat{T} \setminus \{0\}} \mathbb{P}R_\xi.$$

A careful exposition and proof of the identification (13) is the purpose of the following Section 3 (see Proposition 1). It is worth recalling that the elements of $\text{Inv}(X, G)$ are really effective divisors; to keep them reduced, we would need to remove classes of non-reduced elements of R from the right hand side of (13). Since we are interested in lowest degree elements, thus automatically reduced, the distinction is irrelevant for our purposes.

Our main result, Theorem 1, is recast in this language as Theorem 2 in Section 4. Somewhat more can be said in special cases (see Proposition 2). For \mathfrak{g} of type A, the ring R is freely generated by the pair of elements of degree 1. These are interchanged by the outer automorphism of G corresponding to the reflection symmetry of the Dynkin diagram (i.e. transposition of matrices); their product (corresponding to a union of hypersurfaces) is the only reduced element invariant under the full automorphism group of \mathfrak{g} . For \mathfrak{g} of type G_2 , the ring R is generated by a single element of degree 3. Finally, let us remark that for \mathfrak{g} of type B and D, as well as A and G_2 , we construct explicit forms of the lowest degree non-constant elements of R . This allows us to write down the corresponding PDEs in coordinates (see Section 6).

3. PREREQUISITES

3.1. The adjoint variety as a contact manifold. We will now return to the notions and constructions introduced informally in Section 2 above, providing necessary definitions and proving some of their properties. The results are standard, but we give detailed proofs wherever they contain some valuable insights for the non-expert reader. Some of the material of the previous sections is repeated in order to make the present one self-contained. Our ultimate goal is Proposition 1. The reader is invited to skip directly to the latter should the exposition become too pedagogical.

We fix \mathfrak{g} , $G \subset \text{Aut } \mathfrak{g}$ and $X \subset \mathbb{P}\mathfrak{g}$ as in Definition 1. Note that G is a connected simple linear algebraic group and X a projective homogeneous variety for G , whence it follows that the stabiliser in G of any point of X is a *parabolic* subgroup. Recall that the structure of such subgroups is easily described in terms of root system data; we will return to this point later (see the proof of Lemma 3).

Let us refresh the notations introduced earlier in Subsection 2.1. The point $o \in X$ is the *origin*, and $P \subset G$ is its stabiliser. The normal subgroup $P_+ \subset P$, consisting of the unipotent elements in the radical, is the *unipotent radical* of P . The quotient group $G_0 = P/P_+$ is reductive, namely a product of a connected semisimple group G_0^{ss} and a central torus T (note that in our setting a torus means a direct product of several copies of the multiplicative group \mathbb{C}^\times). Later we will determine the type of $\mathfrak{g}_0^{\text{ss}}$ and the rank of T in terms of root system data (see the table in Lemma 3). Symbols $\mathfrak{p}_+ \subset \mathfrak{p} \subset \mathfrak{g}$ denotes the Lie algebras of $P_+ \subset P \subset G$.

Now we define a natural P -invariant *decreasing filtration* on \mathfrak{g} , by setting $\mathfrak{g}^{i+1} = [\mathfrak{p}_+, \mathfrak{g}^i]$ unless $\mathfrak{g}^{i+1} = \mathfrak{g}$ and arranging the indices so that $\mathfrak{g}^0 = \mathfrak{p}$. Lemma 1 below captures the key properties of this filtration. To this end, denote by $\text{gr}_\bullet(\mathfrak{g}/\mathfrak{p})$ the associated graded space of the induced filtration on $\mathfrak{g}/\mathfrak{p}$, with the natural action of P .

Lemma 1. *Let \mathfrak{g}^\bullet be the filtration on \mathfrak{g} satisfying $\mathfrak{g}^0 = \mathfrak{p}$ and $\mathfrak{g}^{i+1} = [\mathfrak{p}_+, \mathfrak{g}^i]$ unless $\mathfrak{g}^{i+1} = \mathfrak{g}$. Then:*

- (1) $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$,
- (2) $\mathfrak{g}^{-i} = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = 0$ for all $i \geq 2$,
- (3) *there is an induced P -invariant filtration on $\mathfrak{g}/\mathfrak{p}$.*
- (4) P_+ *acts trivially on $\text{gr}(\mathfrak{g}/\mathfrak{p})$,*
- (5) $\text{gr}(\mathfrak{g}/\mathfrak{p}) = \text{gr}_{-2}(\mathfrak{g}/\mathfrak{p}) \oplus \text{gr}_{-1}(\mathfrak{g}/\mathfrak{p})$,
- (6) $\dim \text{gr}_{-2}(\mathfrak{g}/\mathfrak{p}) = 1$, $\dim \text{gr}_{-1}(\mathfrak{g}/\mathfrak{p})$ *is even, and the map*

$$\omega : \Lambda^2 \text{gr}_{-1}(\mathfrak{g}/\mathfrak{p}) \rightarrow \text{gr}_{-2}(\mathfrak{g}/\mathfrak{p})$$

*induced by the Lie bracket is a P -equivariant twisted symplectic form.*²

We will prove Lemma 1 once we introduce a *grading* on \mathfrak{g} splitting the filtration, and describe it in terms of the root system data, i.e., after the proof of Lemma 3. The reason for delaying this step is that the filtration is natural, i.e., it is well-defined as soon as we have chosen the origin $o \in X$ (equivalently the parabolic $P \subset G$), while the grading

²The condition of ω being a twisted symplectic form means that the induced map $\text{gr}_{-1}(\mathfrak{g}/\mathfrak{p}) \rightarrow \text{gr}_{-1}(\mathfrak{g}/\mathfrak{p})^* \otimes \text{gr}_{-2}(\mathfrak{g}/\mathfrak{p})$ is an isomorphism.

will require us to make an additional choice (corresponding to fixing a homomorphism $G_0 \hookrightarrow P$ splitting the natural projection). Of course, passing to the root system description requires even further choices.

Let us now identify X with G/P , so that the coset gP corresponds to $g \cdot o$. Viewing $G \rightarrow X$ as a P -principal bundle, we may then identify the tangent bundle of X with an associated bundle:

$$TX \simeq G \times^P (\mathfrak{g}/\mathfrak{p}).$$

The P -equivariant filtration on $\mathfrak{g}/\mathfrak{p}$ induces then a G -invariant filtration on TX . The only non-trivial sub-bundle we obtain this way is

$$TX \supset \mathcal{C} \simeq G \times^P (\mathfrak{g}^{-1}/\mathfrak{p}) \subset G \times^P (\mathfrak{g}/\mathfrak{p}).$$

Lemma 2. *The sub-bundle $\mathcal{C} \subset TX$ is a G -invariant contact distribution. Furthermore, the Levi bracket $\Lambda^2 \mathcal{C} \rightarrow TX/\mathcal{C}$ evaluated at $o \in X$ coincides with the map ω under the identifications $\mathcal{C}_o \simeq \text{gr}_{-1}(\mathfrak{g}/\mathfrak{p})$ and $T_o X/\mathcal{C}_o \simeq \text{gr}_{-2}(\mathfrak{g}/\mathfrak{p})$.*

Proof. We use Lemma 1, in particular (5) and (6). G -invariance of \mathcal{C} follows from P -invariance of \mathfrak{g}^{-1} . Furthermore, we have a natural identification

$$TX/\mathcal{C} \simeq G \times^P \text{gr}_{-2}(\mathfrak{g}/\mathfrak{p})$$

so that in particular \mathcal{C} has corank 1 in TX . To check that \mathcal{C} is a contact distribution it will be enough to show commutativity of the following diagram of vector bundle homomorphisms:

$$\begin{array}{ccc} \Lambda^2 \mathcal{C} & \longrightarrow & TX/\mathcal{C} \\ \parallel & & \parallel \\ G \times^P \Lambda^2 \text{gr}_{-1}(\mathfrak{g}/\mathfrak{p}) & \xrightarrow{G \times^P \omega} & G \times^P \text{gr}_{-2}(\mathfrak{g}/\mathfrak{p}) \end{array}$$

where the top horizontal arrow is the Levi bracket, and the vertical arrows are the natural identifications with associated bundles. By G -equivariance, it is enough to restrict to fibres over $o \in X$, where we want to show that the Levi bracket $\Lambda^2 \mathcal{C}_o \rightarrow T_o X/\mathcal{C}_o$ corresponds to ω under the identification $T_o X \simeq \mathfrak{g}/\mathfrak{p}$ and $\mathcal{C}_o \simeq \mathfrak{g}^{-1}/\mathfrak{p}$. This is clear once one considers the commutative diagram

$$\begin{array}{ccccc} \mathfrak{g} & \longrightarrow & \Gamma(X, TX) & & \mathfrak{g}^{-1} & \longrightarrow & \Gamma(X, \mathcal{C}) \\ \downarrow & & \downarrow & \text{restricting to} & \downarrow & & \downarrow \\ \mathfrak{g}/\mathfrak{p} & \xlongequal{\quad} & T_o X & & \mathfrak{g}^{-1}/\mathfrak{p} & \xlongequal{\quad} & \mathcal{C}_o \end{array}$$

where the top horizontal arrow is the infinitesimal action of \mathfrak{g} on X , the left vertical arrow is the natural projection, and the right vertical one is the evaluation at the origin. \square

The construction of the contact structure we have given here emphasises the homogeneous space aspect of X . One may approach it from a different angle as well: as the contact projectivisation of a symplectic orbit of G in \mathfrak{g} . We will not pursue this interpretation.

3.2. The contact grading and the root system data. It is standard—and notationally convenient—to work with a grading on \mathfrak{g} rather than a filtration. We will follow this custom in the remainder of this article; the previous subsection served as the last reminder that the filtration is geometrically more fundamental. Recall that the parabolic $P \subset G$ may be identified (non-canonically) with the semi-direct product (3), where $G_0 = P/P_+$ is the Levi factor. Let us now fix one such identification, amounting to choosing a homomorphism $G_0 \rightarrow P$ splitting the natural projection, and from now on view G_0 as a subgroup of P (and thus also of G). Now, since G_0 is *reductive*, it follows that each $\mathfrak{g}^{i+1} \subset \mathfrak{g}^i$ has a G_0 -invariant complement \mathfrak{g}_i in \mathfrak{g}^i , and we obtain a G_0 -equivariant vector space decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$.

Lemma 3. *The G_0 -equivariant decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ splitting the filtration \mathfrak{g}^\bullet is unique, and satisfies the following properties:*

- (1) $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$,
- (2) $\mathfrak{g}_{-i} = \mathfrak{g}_i = 0$ for $i > 2$,
- (3) $\mathfrak{g}_{-i}^* \simeq \mathfrak{g}_i$ as representations of G_0 ,
- (4) $\dim \mathfrak{g}_{-2} = 1$, $\dim \mathfrak{g}_{-1}$ is even, and the map

$$(14) \quad \omega : \Lambda^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$$

induced by the Lie bracket is a G_0 -equivariant twisted symplectic form.

Furthermore, recalling the decomposition $G_0 = G_0^{ss} \times T$, the following table lists, for the different Cartan types of \mathfrak{g} , the isomorphism type of \mathfrak{g}_0^{ss} , the rank of T , and \mathfrak{g}_{-1} as a representation of \mathfrak{g}_0^{ss} .

restriction	type of \mathfrak{g}	\mathfrak{g}_0^{ss}	$\text{rk } T$	\mathfrak{g}_{-1}
$n \geq 1$	A_{n+1}	\mathfrak{sl}_n	2	$\mathbb{C}^n \oplus \mathbb{C}^{n*}$
$n \geq 3$	$B_{(n+3)/2}$ or $D_{(n+4)/2}$	$\mathfrak{sl}_2 \oplus \mathfrak{so}_n$	1	$\mathbb{C}^2 \otimes \mathbb{C}^n$
$n \geq 2$	C_{n+1}	\mathfrak{sp}_n	1	\mathbb{C}^{2n}
	E_6	\mathfrak{sl}_6	1	$\Lambda^3 \mathbb{C}^6$
	E_7	\mathfrak{spin}_{12}	1	spinor
	E_8	E_7	1	fundamental 56-dim
	F_4	\mathfrak{sp}_3	1	$\Lambda_0^3 \mathbb{C}^6$
	G_2	\mathfrak{sl}_2	1	$S^3 \mathbb{C}^2$

We refer to the above grading as *the contact grading* of \mathfrak{g} (it is precisely the grading (4), see Subsection 2.1). The proof uses the structure theory of \mathfrak{g} . We will only introduce it locally, as it will not be needed throughout most of the paper—until it reappears in Subsection 4.6, where we supplement it with further representation-theoretic entities.

Proof. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and let $\Phi \subset \mathfrak{h}^*$ be the root system of \mathfrak{g} . It is sometimes useful to consider also the maximal torus $H \subset G$ corresponding to \mathfrak{h} , and its lattice of characters; since G is of adjoint type, that lattice may be identified with the root lattice generated in \mathfrak{h}^* by Φ . Write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

for the root space decomposition. Let us further choose a subset $\Phi^+ \subset \Phi$ of positive roots, and a system $\Delta \subset \Phi^+$ of simple roots. Given an element $\alpha \in \mathbb{Z}\Phi$ of the root lattice, write $\alpha > 0$ if $\alpha \in \mathbb{Z}_+ \Delta$. The direct sum of \mathfrak{h} and root subspaces \mathfrak{g}_α , $\alpha > 0$ is a Borel subalgebra $\mathfrak{b} \subset \mathfrak{p}$. Let $\gamma \in \Phi^+$ be the *longest root*, i.e. the highest weight of the adjoint representation \mathfrak{g} . The one-dimensional subspace $\mathfrak{g}_\gamma \subset \mathfrak{g}$ defines a point in $\mathbb{P}\mathfrak{g}$; furthermore, since the G -orbit of this point is closed, we have that in fact \mathfrak{g}_γ belongs to the adjoint variety X . Possibly conjugating the choices we've made thus far, we shall assume that \mathfrak{g}_γ is the origin $o \in X$. We may now compute $\mathfrak{p} \subset \mathfrak{g}$ as the subalgebra stabilising $\mathfrak{g}_\gamma \subset \mathfrak{g}$: being H -invariant, it is a direct sum of \mathfrak{h} and some root subspaces. Clearly, \mathfrak{p} contains \mathfrak{b} ; on the other hand, $\mathfrak{g}_{-\alpha}$, $\alpha > 0$ stabilises \mathfrak{g}_γ if and only if α is orthogonal to γ . We thus find \mathfrak{p} and its nilpotent radical \mathfrak{p}_+ to be:

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha \oplus \bigoplus_{\substack{\alpha > 0 \\ \langle \alpha, \gamma \rangle = 0}} \mathfrak{g}_{-\alpha}, \quad \mathfrak{p}_+ = \bigoplus_{\substack{\alpha > 0 \\ \langle \alpha, \gamma \rangle \neq 0}} \mathfrak{g}_\alpha$$

where (\cdot, \cdot) denotes the inner product on \mathfrak{h}^* induced by the Killing form. Again by the freedom to conjugate our choices by an element of P , we may assume H is contained in $G_0 \subset P$; then, being H -invariant, \mathfrak{g}_0 is necessarily a direct sum of root subspaces and we find:

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\langle \alpha, \gamma \rangle = 0} \mathfrak{g}_\alpha.$$

In particular, $\mathfrak{h} \subset \mathfrak{g}_0$ is a Cartan subalgebra, and \mathfrak{g}_0 is a reductive Lie algebra with root system $\Phi_0 \subset \Phi$ consisting of roots orthogonal to γ . It is at this point clear that the G_0 -invariant splitting of the filtration \mathfrak{g}^\bullet is unique, since: i) the resulting \mathfrak{g}_i are necessarily direct sums of root subspaces and subspaces of \mathfrak{h} , ii) \mathfrak{h} must be entirely contained in \mathfrak{g}_0 , for it is contained in \mathfrak{p} , and has trivial intersection with $\mathfrak{p}_+ = [\mathfrak{p}, \mathfrak{p}_+]$. It is enough to exhibit such a splitting. Let $\gamma^\vee \in \mathfrak{h}$ denote the coroot corresponding to γ , and let $\langle \alpha, \gamma^\vee \rangle$ denote the Cartan pairing for a root α . We then set³

$$\mathfrak{g}_i = \bigoplus_{\langle \alpha, \gamma^\vee \rangle = i} \mathfrak{g}_\alpha \oplus \begin{cases} \mathfrak{h} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

and observe that

$$\mathfrak{g}_{-2} = \mathfrak{g}_{-\gamma}, \quad \mathfrak{g}_{-1} = \bigoplus_{\substack{\alpha > 0 \\ \langle \alpha, \gamma \rangle \neq 0}} \mathfrak{g}_{-\alpha}, \quad \mathfrak{g}_0 = \bigoplus_{\langle \alpha, \gamma \rangle = 0} \mathfrak{g}_\alpha \oplus \mathfrak{h}, \quad \mathfrak{g}_1 = \bigoplus_{\substack{\alpha > 0 \\ \langle \alpha, \gamma \rangle \neq 0}} \mathfrak{g}_\alpha, \quad \mathfrak{g}_2 = \mathfrak{g}_\gamma.$$

In particular, \mathfrak{g}_0 above is the Lie algebra of $G_0 \subset P$, justifying the notation. By construction, $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ is a grading compatible with the Lie bracket, and setting $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$ we have $\mathfrak{g}^0 = \mathfrak{p}$, $\mathfrak{g}^1 = \mathfrak{p}_+ = [\mathfrak{p}_+, \mathfrak{p}]$, $\mathfrak{g}^2 = [\mathfrak{p}_+, \mathfrak{p}_+]$. Using the fact that the roots of \mathfrak{g}_1 are non-orthogonal to γ , it is also not difficult to check that the bracket map $\mathfrak{g}_1 \otimes \mathfrak{g}_i \rightarrow \mathfrak{g}_{i+1}$ is surjective for all $2 \leq i \leq 2$, thus proving that $[\mathfrak{g}^i, \mathfrak{p}_+] = \mathfrak{g}^{i+1}$. Thus the grading defined using γ^\vee does indeed split the filtration induced by \mathfrak{p}_+ .

We have thus established uniqueness, and found an explicit description of the grading. Claims (1) and (2) are immediate. Claim (3) is straightforward, since \mathfrak{g}_0 is clearly self-dual via the Killing form, while the root subspaces in \mathfrak{g}_i and \mathfrak{g}_{-i} for $i \neq 0$ correspond to opposite subsets of Φ . To prove (4), observe that for every positive root α with $\langle \alpha, \gamma^\vee \rangle = 1$, the vector $\gamma - \alpha$ is also a positive root, and $\langle \gamma - \alpha, \gamma^\vee \rangle = 2 - 1 = 1$. Hence the map $\omega : \Lambda^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ decomposes into a sum of isomorphisms $\mathfrak{g}_{-\alpha} \otimes \mathfrak{g}_{-\gamma+\alpha} \rightarrow \mathfrak{g}_{-\gamma}$.

It remains to verify the entries in the table. This is done on a case-by-case basis, and we shall only sketch the argument. Since \mathfrak{g}_0^{ss} is a semi-simple Lie algebra with root system Φ_0 , its Cartan type may be encoded by its Dynkin diagram. Then, since $\Delta_0 = \Delta \cap \Phi_0$ provides a system of simple roots for Φ_0 , we find that the Dynkin diagram of \mathfrak{g}_0^{ss}

³In other words, $\gamma^\vee \in \mathfrak{h}$ serves as the so-called *grading element*.

is a sub-diagram of the Dynkin diagram of \mathfrak{g} , obtained by removing the nodes corresponding to fundamental weights entering with non-zero coefficients into the highest weight of the adjoint representation. At the same time, it follows that the Cartan subalgebra \mathfrak{h}_0 of $\mathfrak{g}_0^{\text{ss}}$ has rank equal to the cardinality of Δ_0 ; since the Lie algebra of the torus T provides a complement to \mathfrak{h}_0 in \mathfrak{h} , it follows that $\text{rk } T$ is equal to the number of removed nodes. Finally, in order to find the decomposition of \mathfrak{g}_{-1} into irreducible $U(\mathfrak{g}_0^{\text{ss}})$ -modules,⁴ it is enough to find roots $\alpha > 0$ with $\langle \alpha, \gamma^\vee \rangle = 1$ and such that $\alpha - \beta \notin \Phi$ for all $\beta \in \Phi_0 \cap \Phi^+$. Then, to each such root α corresponds an irreducible summand of \mathfrak{g}_{-1} whose highest weight with respect to \mathfrak{h}_0 is the image of $-\alpha$ under the natural projection $\mathfrak{h}^* \rightarrow \mathfrak{h}_0^*$. Now, such roots α are precisely the simple roots in $\Delta \setminus \Delta_0$, and to each $\alpha \in \Delta \setminus \Delta_0$ we have the highest weight $U(\mathfrak{g}_0^{\text{ss}})$ -module with highest weight $\lambda(\alpha)$. Evaluating the latter on a coroot associated with a simple root $\beta \in \Delta_0$ of $\mathfrak{g}_0^{\text{ss}}$, we have

$$\langle \lambda(\alpha), \beta^\vee \rangle = -\langle \alpha, \beta^\vee \rangle \geq 0$$

so that the coefficients may be read off the Cartan matrix of \mathfrak{g} . Applying this recipe for each type one finds that

- in type A_{n+1} the set $\Delta \setminus \Delta_0$ consists of the two extreme nodes of the Dynkin diagram Δ , and the highest weights of the two summands of \mathfrak{g}_{-1} are fundamental, corresponding to the two extreme nodes of the Dynkin subdiagram Δ_0 ;
- in all remaining types the set $\Delta \setminus \Delta_0$ consists of a single node α , and the highest weight of \mathfrak{g}_{-1} is: the fundamental weight corresponding to the node of the Dynkin subdiagram Δ_0 adjacent to α , *times* the number of edges connecting the two if the arrow points away from α .

These translate into the data we have included in the table. \square

It is now easy to prove the properties of the filtration \mathfrak{g}^\bullet used in the previous subsection.

Proof of Lemma 1. Recall that in the proof of Lemma 3 we have identified

$$\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j.$$

Then claim (1) of Lemma 1 follows from claim (1) of Lemma 3, as well as claim (2) of the former from claim (2) of the latter. Claim (3) is then obvious, since $\mathfrak{p} = \mathfrak{g}^0$, and so is claim (4), since $P_+ = \exp \mathfrak{p}_+$. Finally, claims (5) and (6) follow immediately from Lemma 3 once one identifies $\text{gr}(\mathfrak{g}/\mathfrak{p})$ with $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. \square

Notation. From now on we shall use the graded subspaces \mathfrak{g}_i , in particular identifying \mathfrak{g}_i with $\text{gr}_i(\mathfrak{g}/\mathfrak{p})$ for $i < 0$. We thus view \mathfrak{g}_{-1} as a representation of P , with the trivial action of P_+ . Furthermore, we identify the fibre $X_o^{(1)}$ of the Lagrangian Grassmannian bundle at the origin with the Lagrangian Grassmannian $\text{LGr } \mathfrak{g}_{-1}$ of the conformal symplectic space \mathfrak{g}_{-1} . We will alternate between the two notations depending on context.

3.3. The Lagrangian Grassmannian as a homogeneous space. The contact space $\mathcal{C}_o \simeq \mathfrak{g}_{-1}$ at the origin plays the role of a model for the intrinsic geometry of the contact distribution. In particular (see Section 2.1), the bundle of Lagrangian Grassmannians $X^{(1)}$ introduced in (6) is an associated bundle with fibre modelled on the Lagrangian Grassmannian of the twisted symplectic space \mathfrak{g}_{-1} :

$$(15) \quad X^{(1)} \simeq G \times^P \text{LGr } \mathfrak{g}_{-1}.$$

In fact, we shall typically use the notation $X_o^{(1)}$ instead of $\text{LGr } \mathfrak{g}_{-1}$. As we have already observed, the action of P on $X_o^{(1)}$ factors through $G_0 \simeq P/P_+$, whence all problems we consider in this paper may be reduced to the study of the G_0 -action on $X_o^{(1)}$. Before we reach this point, we need to understand $X_o^{(1)}$ as a homogeneous space for the full symplectic group $\text{Sp } \mathfrak{g}_{-1}$. The latter denotes the stabiliser in $\text{GL } \mathfrak{g}_{-1}$ of any symplectic form obtained from ω by trivialising $\mathfrak{g}_{-2} \simeq \mathbb{C}$. Let us first explain the place of G_0 in this picture.

Lemma 4. *Every trivialisation of $\mathfrak{g}_{-2} \simeq \mathbb{C}$ gives rise to a symplectic form on \mathfrak{g}_{-1} , and all such forms differ by scaling. The stabiliser of any such form is the same subgroup of $\text{GL } \mathfrak{g}_{-1}$, denoted $\text{Sp } \mathfrak{g}_{-1}$, while the stabiliser of the one-dimensional space of all such forms is denoted $\text{CSp } \mathfrak{g}_{-1} \simeq \text{Sp } \mathfrak{g}_{-1} \times \mathbb{C}^\times$. The natural action of G_0 on \mathfrak{g}_{-1} embeds G_0 as a subgroup of $\text{CSp } \mathfrak{g}_{-1}$ and G_0^{ss} as a subgroup of $\text{Sp } \mathfrak{g}_{-1}$.*

Proof. The only non-trivial statement is that G_0^{ss} is contained in $\text{Sp } \mathfrak{g}_{-1}$. But since the action of G_0^{ss} clearly preserves the one-dimensional space of symplectic forms on \mathfrak{g}_{-1} identified with \mathfrak{g}_{-2}^* , it follows that G_0^{ss} acts on this space by a character $\chi : G_0^{\text{ss}} \rightarrow \mathbb{C}^\times$. Of course, by semi-simplicity, χ is trivial, so that in fact G_0^{ss} preserves every non-zero symplectic form parameterised by \mathfrak{g}_{-2}^* . \square

We shall reserve the symbol n , defined earlier in (5), for the half-dimension of the symplectic space, or contact distribution, throughout the remainder of this paper. We use $\mathfrak{sp}(\mathfrak{g}_{-1})$ to denote the Lie algebra of $\text{Sp } \mathfrak{g}_{-1}$, a subalgebra of $\text{End } \mathfrak{g}_{-1}$.

Definition 2. *The Lagrangian Grassmannian $\text{LGr } \mathfrak{g}_{-1}$ is the submanifold of $\text{Gr}(n, \mathfrak{g}_{-1})$ parameterising maximal isotropic subspaces, i.e., n -dimensional linear subspaces $L \subset \mathfrak{g}_{-1}$ such that $\omega|_L = 0$.*

⁴ U denotes the universal enveloping algebra. We'll sometimes use the language of $U(\mathfrak{g}_0^{\text{ss}})$ -modules instead of representations of $\mathfrak{g}_0^{\text{ss}}$.

It is well-known that the group $\mathrm{Sp} \mathfrak{g}_{-1}$ acts transitively on $\mathrm{LGr} \mathfrak{g}_{-1}$. Given a point in $\mathrm{LGr} \mathfrak{g}_{-1}$ corresponding to $L \subset \mathfrak{g}_{-1}$, its stabiliser in $\mathrm{Sp} \mathfrak{g}_{-1}$ is a maximal parabolic subgroup, as we will soon see in the context of a minimal projective embedding. A more direct description of the stabiliser may be obtained by fixing a Lagrangian complement to L . Such a complement is identified with $L^* \otimes \mathfrak{g}_{-2}$, and the choice of a nonzero element in \mathfrak{g}_{-2} further identifies it with L^* . As this involves arbitrary choices, we will avoid its use in what follows; it is however convenient for concrete computations. The proof is completely standard, and thus omitted.

Lemma 5. *The choice of a bi-Lagrangian decomposition*

$$\mathfrak{g}_{-1} = L \oplus L^*$$

induces:

- (1) *a graded decomposition*

$$\mathfrak{sp}(\mathfrak{g}_{-1}) = S^2 L^* \oplus \mathrm{End} L \oplus S^2 L$$

with $\mathrm{End} L$ in degree 0, acting naturally on the remaining two summands in degrees ± 1 , and with $[\varphi, \psi] = \psi \circ \varphi$ for $\varphi \in S^2 L^*$, $\psi \in S^2 L$ viewed as maps $\varphi : L \rightarrow L^*$, $\psi : L^* \rightarrow L$;

- (2) *identifications*

$$\mathrm{Stab} L = \mathrm{GL} L \ltimes S^2 L, \quad \mathrm{Stab} L^* = S^2 L^* \ltimes \mathrm{GL} L, \quad \mathrm{Stab} L \cap \mathrm{Stab} L^* = \mathrm{GL} L$$

where $\mathrm{Stab} L$, $\mathrm{Stab} L^*$ are the stabilisers of L , L^* in $\mathrm{Sp} \mathfrak{g}_{-1}$, and we view $S^2 L$, $S^2 L^*$ as vector groups;

- (3) *a $\mathrm{GL} L$ -equivariant identification*

$$S^2 L^* \simeq \{\Lambda \in \mathrm{LGr} \mathfrak{g}_{-1} \mid \Lambda \cap L^* = 0\}$$

sending $\varphi \in S^2 L^*$ to the graph of $\varphi : L \rightarrow L^*$.

We remark that, from the point of view of a G_0 -action, a natural bi-Lagrangian decomposition of the symplectic vector space \mathfrak{g}_{-1} exists only in type A, where G_0^{ss} is precisely the semisimple part of the Levi factor GL_n of the parabolic arising as a stabiliser of a point of $\mathrm{LGr} \mathfrak{g}_{-1}$ in $\mathrm{Sp} \mathfrak{g}_{-1}$.

3.4. The Plücker embedding. The Lagrangian Grassmannian comes equipped with a distinguished $\mathrm{Sp} \mathfrak{g}_{-1}$ -equivariant embedding into the projectivisation of an irreducible representation (more precisely, the kernel of the map (7), see Subsection 2.1). We will describe its properties in this subsection, along with some further data on the representation theory of the symplectic group. As before, we confine our use of structure theory to the proofs.

Definition 3. *Let $\dim \mathfrak{g}_{-1} = 2n$, as in (5).*

- (1) $\Lambda_0^i \mathfrak{g}_{-1}$ denotes the kernel of the map $\Lambda^i \mathfrak{g}_{-1} \rightarrow \Lambda^{i-2} \otimes \mathfrak{g}_{-2}$ given by contraction with $\omega \in \Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}$,
- (2) $\Lambda_0^n \mathfrak{g}_{-1}$ is called the (Lagrangian) Plücker space (cf. (7)).

For $i = 1$ we understand the above map to be zero, so that $\Lambda_0^i \mathfrak{g}_{-1} = \Lambda^i \mathfrak{g}_{-1}$ by definition.

Lemma 6.

- (1) *The spaces $\Lambda_0^i \mathfrak{g}_{-1}$, $1 \leq i \leq n$ are precisely the fundamental irreducible representations of $\mathrm{Sp} \mathfrak{g}_{-1}$.*
- (2) *For each $d \geq 0$ the symmetric power $S^d \Lambda_0^n \mathfrak{g}_{-1}$ contains an irreducible subrepresentation $S_0^d \Lambda_0^n \mathfrak{g}_{-1}$ spanned by elements of the form $(\det L)^d$ for $L \subset \mathfrak{g}_{-1}$ a Lagrangian subspace.*

Proof. A proof of part (1) may be found in [5, Ch. VIII, 3]; we quote some of its points. If we fix a bi-Lagrangian decomposition $\mathfrak{g}_{-1} = L \oplus L^*$ inducing an embedding $\mathrm{End} L \subset \mathfrak{sp}(\mathfrak{g}_{-1})$, every Cartan subalgebra of $\mathrm{End} L$ is also a Cartan subalgebra of $\mathfrak{sp}(\mathfrak{g}_{-1})$. Fixing a basis $e_1, \dots, e_n \in L$, let $\mathfrak{h} \subset \mathrm{End} L$ be the corresponding diagonal subalgebra, with a basis H_1, \dots, H_n such that $H_i(e_j) = \delta_{ij} e_j$. Letting $\eta_1, \dots, \eta_n \in \mathfrak{h}^*$ be the dual basis, it turns out that we can write a system of simple roots as

$$\alpha_i = \eta_i - \eta_{i+1}, \quad 1 \leq i \leq n-1, \quad \alpha_n = 2\eta_n$$

with corresponding fundamental weights

$$\lambda_i = \eta_1 + \dots + \eta_i, \quad 1 \leq i \leq n.$$

Then $e_1 \wedge \dots \wedge e_i$ is the highest weight vector in $\Lambda_0^i \mathfrak{g}_{-1}$, with weight λ_i . In particular $\det L$ is the highest weight vector in $\Lambda_0^n \mathfrak{g}_{-1}$, so that $(\det L)^d$ is the highest weight vector of some irreducible subrepresentation $V \subset S^d \Lambda_0^n \mathfrak{g}_{-1}$. Since $\mathrm{Sp} \mathfrak{g}_{-1}$ acts transitively on $\mathrm{LGr} \mathfrak{g}_{-1}$, it follows that $(\det L')^d \in V$ as well for every other Lagrangian n -plane $L' \subset \mathfrak{g}_{-1}$. Finally, since the span of all such elements is a sub-representation, it must coincide with V by irreducibility, thus proving (2). \square

Lemma 7. *Sending a Lagrangian subspace $L \subset \mathfrak{g}_{-1}$ to its determinant $\det L \in \Lambda_0^n \mathfrak{g}_{-1}$ defines an $\mathrm{Sp} \mathfrak{g}_{-1}$ -equivariant projective embedding*

$$(16) \quad \iota : \mathrm{LGr} \mathfrak{g}_{-1} \rightarrow \mathbb{P} \Lambda_0^n \mathfrak{g}_{-1}$$

onto the unique closed $\mathrm{Sp} \mathfrak{g}_{-1}$ -orbit. Furthermore, for each $d \geq 0$ the natural map

$$S^d \Lambda_0^n \mathfrak{g}_{-1}^* \rightarrow \Gamma(\mathrm{LGr} \mathfrak{g}_{-1}, \iota^* \mathcal{O}(d))$$

is $\mathrm{Sp} \mathfrak{g}_{-1}$ -equivariant and factors through an isomorphism with $S_0^d \Lambda_0^n \mathfrak{g}_{-1}^*$.

Let us first explain that given a point of $\mathrm{LGr} \mathfrak{g}_{-1}$ corresponding to a Lagrangian $L \subset \mathfrak{g}_{-1}$, the fibre of $\iota^* \mathcal{O}(d)$ consists of degree d homogeneous polynomials on the rank one vector space $\det L \subset \Lambda_0^n \mathfrak{g}_{-1}$, and thus identifies with the dual of $(\det L)^d \subset S^d \Lambda_0^n \mathfrak{g}_{-1}$. Hence, every element of $S^d \Lambda_0^n \mathfrak{g}_{-1}^*$ defines by restriction an element in the fibre of $\iota^* \mathcal{O}(d)$ over any point of $\mathrm{LGr} \mathfrak{g}_{-1}$, thus giving rise to a global section. Factorisation is then immediate, by point (2) of Lemma 6. The non-trivial statement is that we do get an isomorphism.

Proof. We use the setup introduced in the proof of Lemma 6. We know that $\Lambda_0^n \mathfrak{g}_{-1}$ is the irreducible representation with highest weight λ_n , and $\det L$ is the highest weight line, where $L = \langle e_1, \dots, e_n \rangle$. The stabiliser of $\det L$ is a fundamental parabolic subgroup $Q \subset \mathrm{Sp} \mathfrak{g}_{-1}$ with Lie algebra \mathfrak{q} whose Levi factor \mathfrak{q}_0 is a reductive Lie algebra with \mathfrak{h} as a Cartan subalgebra, and a root system generated by the simple roots $\alpha_1, \dots, \alpha_{n-1}$. In particular, the simple highest weight $U(\mathfrak{q}_0)$ -module with highest weight $d\lambda_n$, $d \geq 0$, is one-dimensional, and we denote it by $\mathbb{C}_{d\lambda_n}$. Letting the unipotent radical act trivially, we inflate it to a $U(\mathfrak{q})$ -module, and furthermore view it as a representation of Q (note that the maximal torus in the Levi factor Q_0 corresponding to \mathfrak{h} is the same as in the simply-connected group $\mathrm{Sp} \mathfrak{g}_{-1}$, so its character lattice coincides with the full integral weight lattice, in particular containing $d\lambda_n$).

Now, since ι is well-defined and $\mathrm{Sp} \mathfrak{g}_{-1}$ -equivariant, it maps $\mathrm{LGr} \mathfrak{g}_{-1}$ onto the highest weight orbit $\mathrm{Sp} \mathfrak{g}_{-1}/Q$ in $\mathbb{P}\Lambda_0^n \mathfrak{g}_{-1}$. Furthermore, since L is precisely the space of vectors $v \in \mathfrak{g}_{-1}$ such that $v \wedge \det L = 0$, the map ι is injective. We may identify $\iota^* \mathcal{O}(d)$ with the associated bundle $\mathrm{Sp} \mathfrak{g}_{-1} \times^Q \mathbb{C}_{d\lambda_n}$, and then it follows from the Borel–Weil theorem that the space of its global sections is isomorphic, as a representation of $\mathrm{Sp} \mathfrak{g}_{-1}$, to the irreducible representation with highest weight $d\lambda_n$, i.e., $S_0^d \Lambda_0^n \mathfrak{g}_{-1}^*$. \square

In particular, for $d = 1$ we find that the natural map discussed above gives a bijection between the dual of the embedding space $\Lambda_0^n \mathfrak{g}_{-1}^*$ and the space of global sections of $\iota^* \mathcal{O}(1)$. The standard terminology is: *linear non-degeneracy* for injectivity at $d = 1$, and *projective normality* for surjectivity at $d > 0$.

Corollary 1. *The Plücker embedding ι is linearly non-degenerate and projectively normal.*

Let us also observe that the kernel of the natural map $S^d \Lambda_0^n \mathfrak{g}_{-1} \rightarrow \Gamma(\mathrm{LGr} \mathfrak{g}_{-1}, \iota^* \mathcal{O}(d))$ consists of homogeneous degree d polynomials on the Plücker space which vanish on $\iota(\mathrm{LGr} \mathfrak{g}_{-1})$ (or, more precisely, on its affine cone). We denote this kernel by I_d . The direct sum

$$(17) \quad I = \bigoplus_d I_d$$

forms the homogeneous ideal of $\iota(\mathrm{LGr} \mathfrak{g}_{-1})$ in $S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^*$.

Corollary 2. *For each $d \geq 0$ we have natural $\mathrm{Sp} \mathfrak{g}_{-1}$ -equivariant isomorphisms*

$$\Gamma(\mathrm{LGr} \mathfrak{g}_{-1}, \iota^* \mathcal{O}(d)) \simeq S_0^d \Lambda_0^n \mathfrak{g}_{-1}^* \simeq S^d \Lambda_0^n \mathfrak{g}_{-1}^* / I_d$$

where I_d is the degree d subspace of the homogeneous ideal of $\iota(\mathrm{LGr} \mathfrak{g}_{-1})$.

As we have already remarked, none of the above requires us to work with a bi-Lagrangian decomposition of \mathfrak{g}_{-1} . Nevertheless, it is sometimes convenient to fix one for computational purposes. The additional structure it induces is summed up in the following Lemma. We omit its proof, since it is straightforward and not essential for our purpose.

Lemma 8. *The choice of a bi-Lagrangian decomposition as in Lemma 5 induces an identification*

$$(18) \quad \Lambda_0^n \mathfrak{g}_{-1} \simeq \bigoplus_{i=0}^n S_0^2 \Lambda^i L^*$$

equivariant under $\mathrm{SL} L \subset \mathrm{Sp} \mathfrak{g}_{-1}$. Furthermore, the restriction of the Plücker embedding (16) to the $\mathrm{GL} L$ -invariant dense open subset defined in Lemma 5 is

$$(19) \quad \iota : S^2 L^* \rightarrow \mathbb{P}\Lambda_0^n \mathfrak{g}_{-1}, \quad \iota(\varphi) = [1 : \varphi : \Lambda^2 \varphi : \dots : \Lambda^{n-1} \varphi : \det \varphi]$$

where $\Lambda^i \varphi \in S_0^2 \Lambda^i L^*$ may be viewed as the matrix of i^{th} minors of $\varphi \in S^2 L^*$.

3.5. Hypersurfaces and invariants. We will now discuss invariant hypersurfaces in $\mathrm{LGr} \mathfrak{g}_{-1}$ and their relation to G_0^{ss} -invariant elements in $S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^*$. The passage to invariants is most natural if one works with *effective divisors* on $\mathrm{LGr} \mathfrak{g}_{-1}$ instead of one-codimensional submanifolds. As the term may not be completely familiar to the broad differential-geometric audience, we shall instead repurpose the term *hypersurface*.

Definition 4.

- (1) A hypersurface in $\mathrm{LGr} \mathfrak{g}_{-1}$ is a finite formal combination $\sum m_i Z_i$ where $Z_i \subset \mathrm{LGr} \mathfrak{g}_{-1}$ are closed, irreducible, one-codimensional subvarieties, and m_i are positive integers.
- (2) The hypersurface cut out by a section $f \in \Gamma(\mathrm{LGr} \mathfrak{g}_{-1}, \iota^* \mathcal{O}(d))$, $f \neq 0$, $d > 0$ is $\sum m_i Z_i$ where the Z_i are the irreducible components of the zero locus of f , while m_i is the order of vanishing of f at a general point of Z_i .

The following fact justifies our choices. Being entirely standard, we just sketch its proof to the reader convenience.

Lemma 9. *The set of hypersurfaces in $\mathrm{LGr} \mathfrak{g}_{-1}$ is in one-to-one correspondence with the disjoint union of $\mathbb{P}\Gamma(\mathrm{LGr} \mathfrak{g}_{-1}, \iota^* \mathcal{O}(d))$ for all $d > 0$.*

Proof. Given a hypersurface $\sum m_i Z_i$, we can find an open cover $\mathrm{LGr} \mathfrak{g}_{-1} = \bigcup U_\alpha$ and meromorphic functions f_α such that f_α is analytic on U_α , vanishes precisely to order m_i at a general point of $Z_i \cap U_\alpha$, and the zeros of $f_\alpha|_{U_\alpha}$ are contained in $\bigcup Z_i$. Then the transition functions $(f_\alpha/f_\beta)|_{U_\alpha \cap U_\beta}$ define a Čech cocycle of invertible analytic functions, and thus a line bundle \mathcal{L} . The collection $f_\alpha|_{U_\alpha}$ may be then viewed as defining an element $f \in \Gamma(\mathrm{LGr} \mathfrak{g}_{-1}, \mathcal{L})$. It is easy to see that any other choice of a covering and transition functions would lead to the same class in $H^1(\mathrm{LGr} \mathfrak{g}_{-1}, \mathcal{O}^\times)$, and thus to an isomorphic line bundle; furthermore, two sections f, g of \mathcal{L} cutting out the same hypersurface give rise to a global analytic function f/g , necessarily constant.

It remains to show that every line bundle over $\mathrm{LGr} \mathfrak{g}_{-1}$ is isomorphic to $\iota^* \mathcal{O}(d)$ for some d . One first checks that for every $g \in \mathrm{Sp} \mathfrak{g}_{-1}$ and every line bundle \mathcal{L} over $\mathrm{LGr} \mathfrak{g}_{-1}$ there is an isomorphism $\phi_g : g^* \mathcal{L} \simeq \mathcal{L}$: since $\mathrm{Sp} \mathfrak{g}_{-1}$ is connected, this follows from discreteness of the Picard group of $\mathrm{LGr} \mathfrak{g}_{-1}$, a consequence of $H^1(\mathrm{LGr} \mathfrak{g}_{-1}, \mathcal{O}) = 0$ as given by the Bott–Borel–Weil Theorem. Then, for each \mathcal{L} one considers the Lie group $H_{\mathcal{L}}$ consisting of pairs (g, ϕ_g) as above, with the obvious multiplication and a forgetful homomorphism $H_{\mathcal{L}} \rightarrow \mathrm{Sp} \mathfrak{g}_{-1}$. This group acts on $\mathrm{LGr} \mathfrak{g}_{-1}$ as well as on \mathcal{L} in the natural way. It is a central extension of $\mathrm{Sp} \mathfrak{g}_{-1}$ by \mathbb{C}^\times , and corresponds infinitesimally to a central extension of Lie algebras. But since $\mathfrak{sp}(\mathfrak{g}_{-1})$ is simple, the latter extension is necessarily split. By simply-connectedness of $\mathrm{Sp} \mathfrak{g}_{-1}$ the splitting homomorphism may be then integrated to $\mathrm{Sp} \mathfrak{g}_{-1} \rightarrow H_{\mathcal{L}}$, providing an action of $\mathrm{Sp} \mathfrak{g}_{-1}$ on \mathcal{L} . This proves that every line bundle over $\mathrm{LGr} \mathfrak{g}_{-1}$ is equivariant, i.e., admits a compatible $\mathrm{Sp} \mathfrak{g}_{-1}$ -action. Finally, via the associated bundle construction, equivariant line bundles are classified up to isomorphism by one-dimensional representations of the parabolic subgroup $Q \subset \mathrm{Sp} \mathfrak{g}_{-1}$ stabilising a point in $\mathrm{LGr} \mathfrak{g}_{-1}$ (as in the proof of Lemma 7), and thus by the characters of the central torus of the Levi factor of Q . Since Q is the stabiliser of the highest weight line in a *fundamental* representation, it follows that the central torus of its Levi factor has rank one. Furthermore, $\iota^* \mathcal{O}(1)$ corresponds to the tautological representation of \mathbb{C}^\times , whence every other equivariant line bundle is its power. \square

Now we formalise properly the notion of *degree*, already discussed in Subsection 2.1, just before the formulation of the main Theorem 1.

Definition 5. A hypersurface in $\mathrm{LGr} \mathfrak{g}_{-1}$ has degree $d > 0$ if it is cut out by a global section of $\iota^* \mathcal{O}(d)$.

Remark 3. Using the description of the Plücker embedding given in Lemma 8, one may verify that Definition 5 above is compatible with the one given informally in Subsection 1.3.

In order to handle the action of $G_0 \subset \mathrm{CSp} \mathfrak{g}_{-1}$ rather than just $G_0^{\mathrm{ss}} \subset \mathrm{Sp} \mathfrak{g}_{-1}$, we need the following facts.

- Lemma 10.** (1) The $\mathrm{Sp} \mathfrak{g}_{-1}$ -action on $\mathrm{LGr} \mathfrak{g}_{-1}$ extends trivially to a $\mathrm{CSp} \mathfrak{g}_{-1}$ -action.
(2) The $\mathrm{CSp} \mathfrak{g}_{-1}$ -action on $\mathrm{LGr} \mathfrak{g}_{-1}$ lifts naturally to an action on $\iota^* \mathcal{O}(d)$ for all d .
(3) The isomorphisms $\Gamma(\mathrm{LGr} \mathfrak{g}_{-1}, \iota^* \mathcal{O}(d)) \simeq S_0^d \Lambda_0^n \mathfrak{g}_{-1}^* \simeq S^d \Lambda_0^n \mathfrak{g}_{-1}^* / I_d$, $d > 0$, are $\mathrm{CSp} \mathfrak{g}_{-1}$ -equivariant.

Proof. Straightforward, given the induced natural action of $\mathrm{CSp} \mathfrak{g}_{-1}$ on the Plücker space $\Lambda_0^n \mathfrak{g}_{-1}$. \square

We are now ready to spell out the invariance condition for a hypersurface in terms of the corresponding section.

Lemma 11. Let $f \in \Gamma(\mathrm{LGr} \mathfrak{g}_{-1}, \iota^* \mathcal{O}(d))$, $f \neq 0$. Then the hypersurface cut out by f is G_0 -invariant if and only if there exists a homomorphism $\xi : G_0 \rightarrow \mathbb{C}^\times$ such that $g^* f = \xi(g) f$ for all $g \in G_0$.

Proof. By Lemma 9 we find that G_0 -invariance of the hypersurface cut out by f is equivalent to the existence, for each $g \in G_0$, of a scaling factor $c_g \in \mathbb{C}^\times$ such that $g^* f = c_g f$. Furthermore, c_g is uniquely determined by g , and we have $c_{gh} f = h^* g^* f = h^* (c_g f) = c_g c_h f$ whence $g \mapsto c_g$ is a character ξ . \square

Recall now the decomposition $G_0 = G_0^{\mathrm{ss}} \times T$, where G_0^{ss} is semi-simple, and T is a torus (cf. 10). It follows that characters of G_0 factor through T , so that f cuts out a G_0 -invariant hypersurface if and only if it is G_0^{ss} -invariant, and transforms under the action of T via some character $\xi \in \widehat{T}$. We now apply the identification spelt out in Corollary 2:

Lemma 12. Let $R = (S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^* / I)^{G_0^{\mathrm{ss}}}$ be the ring of G_0^{ss} -invariants. Then:

- (1) G_0 acts on R , and the action factors through T ,
- (2) $R = \bigoplus_{\xi \in \widehat{T}} R_\xi$ where T acts on R_ξ via $\xi : T \rightarrow \mathbb{C}^\times$,
- (3) the set of G_0 -invariant hypersurfaces in $\mathrm{LGr} \mathfrak{g}_{-1}$ is in one-to-one correspondence with the disjoint union of $\mathbb{P} R_\xi$ for all $\xi \in \widehat{T} \setminus \{0\}$.

Proof. Part (1) follows from centrality of T in G_0 , and part (2) from the finite-dimensionality of the graded summands of R . Part (3) is then a consequence of Corollary 2 and Lemma 11. \square

Let us point out that, since G_0^{ss} is semi-simple, hence reductive, $I_d \subset S^d \Lambda_0^n \mathfrak{g}_{-1}^*$ admits a G_0^{ss} -invariant complement, thus allowing us to identify

$$(20) \quad (S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^* / I)^{G_0^{\mathrm{ss}}} = (S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^*)^{G_0^{\mathrm{ss}}} / I^{G_0^{\mathrm{ss}}}.$$

That is, we may represent a G_0 -invariant hypersurface by a G_0^{ss} -invariant homogeneous polynomial on the Plücker space. Finally we remark that the ‘standard’ grading on R , with $(S^d \Lambda_0^n \mathfrak{g}_{-1}^* / I_d)^{G_0^{\mathrm{ss}}}$ placed in degree d , may be recovered from the rescaled T -weights $\xi \in \widehat{T}$.

Lemma 13. *Let $j : \mathbb{C}^\times \hookrightarrow \mathrm{CSp} \mathfrak{g}_{-1}$ be the one-parameter subgroup acting by scaling on \mathfrak{g}_{-1} . Then j factors through T , inducing a homomorphism $j^* : \widehat{T} \rightarrow \mathbb{Z}$ such that $R_\xi \subset S^d \Lambda_0^n \mathfrak{g}_{-1}^*$ if and only if $j^* \xi = -nd$.*

Proof. Since G_0 contains the full maximal torus of G corresponding to the Cartan subalgebra we used to define the contact grading (4) on \mathfrak{g} , it follows that in particular it contains a one-parameter subgroup acting by scaling on \mathfrak{g}_{-1} , necessarily coinciding with j . Being central, it factors through T . As it acts on $S^d \Lambda_0^n \mathfrak{g}_{-1}^*$ with weight $-nd$, the claim follows. \square

3.6. Back to the adjoint variety. Let us now return to the notion of the adjoint variety X of \mathfrak{g} (recall Definition 1). We have already introduced the bundle

$$X^{(1)} \rightarrow X$$

whose fibre at $x \in X$ is the set of Lagrangian subspaces in \mathcal{C}_x (see Subsection 2.1, (6)), together with the natural G -invariant identification $X^{(1)} \simeq G \times^P \mathrm{LGr} \mathfrak{g}_{-1}$ (see Subsection 3.3, (15)) induced by the identification $\mathcal{C} \simeq G \times^P \mathfrak{g}_{-1}$. We extend our notion of a hypersurface from the fibre $\mathrm{LGr} \mathfrak{g}_{-1} \simeq X_o^{(1)}$ to the entire bundle $X^{(1)}$.

Definition 6.

- (1) *A hypersurface in $X^{(1)}$ is a finite formal combination $\sum m_i Y_i$ where $Y_i \subset X^{(1)}$ are closed, irreducible, codimension 1 subvarieties, and m_i are positive integers.*
- (2) *$\mathrm{Inv}(X, G)$ is the set of G -invariant hypersurfaces in $X^{(1)}$.*

A general hypersurface in $X^{(1)}$ may have components projecting onto a codimension 1 subvariety of X . Clearly, this cannot occur in the G -invariant case, where we do obtain a family of hypersurfaces in the fibres, all conjugate to a single G_0 -invariant hypersurface in $\mathrm{LGr} \mathfrak{g}_{-1}$.

Lemma 14. *Let $\sum m_i Y_i$ be a G -invariant hypersurface in $X^{(1)}$. Then $\sum m_i (Y_i \cap X_o^{(1)})$ is a G_0 -invariant hypersurface in $X_o^{(1)} = \mathrm{LGr} \mathfrak{g}_{-1}$.*

Proof. Let Y be a G -invariant one-codimensional subvariety in $X^{(1)}$. We need to check that $Y \cap X_o^{(1)}$ is a G_0 -invariant codimension 1 subvariety in $X_o^{(1)}$. By G -invariance, the projection $Y \rightarrow X$ is surjective, so that $Y \cap X_x^{(1)}$ is codimension 1 in $X_x^{(1)}$ for general $x \in X$. Again by G -invariance, $Y \cap X_x^{(1)}$ is codimension 1 in $X_x^{(1)}$ for every $x \in X$, in particular for $x = o$. Finally, G_0 -invariance of $Y \cap X_o^{(1)}$ is immediate from G -invariance of Y and P -invariance of o . \square

- Definition 7.** (1) *The fibre at the origin of a G -invariant hypersurface in $X^{(1)}$ is the G_0 -invariant hypersurface in $\mathrm{LGr} \mathfrak{g}_{-1}$ arising as in Lemma 14.*
- (2) *The degree of a G -invariant hypersurface in $X^{(1)}$ is the degree of its fibre at the origin.*

We are now able to exhibit the main result of this long pedagogical section, that is, Proposition 1 below. It finally provides the necessary interpretation of G -invariant hypersurfaces in $X^{(1)}$ in terms of projectivised T -weight subspaces in the ring of G_0^{ss} -invariants in the homogeneous coordinate ring of the Plücker-embedded Lagrangian Grassmannian.

Proposition 1. *There is a natural bijection*

$$\mathrm{Inv}(X, G) \simeq \coprod_{\xi \in \widehat{T} \setminus \{0\}} \mathbb{P}R_\xi.$$

It identifies invariant hypersurfaces of degree $d > 0$ in $X^{(1)}$ with points of the disjoint union of $\mathbb{P}R_\xi$ such that $j^ \xi = -nd$, where $j^* : \widehat{T} \rightarrow \mathbb{Z}$ is the homomorphism of Lemma 13.*

Proof. This follows directly from Lemmas 12, 13 and 14. \square

4. PROOF OF THEOREM 1

4.1. Reformulation. By slight abuse of notation we let $R_d \subset R$ be the degree d component of the ring of G_0^{ss} -invariants in the homogeneous coordinate ring of $\mathrm{LGr} \mathfrak{g}_{-1}$. We then have, according to Lemma 13, a decomposition

$$R_d = \bigoplus_{\substack{\xi \in \widehat{T} \\ j^* \xi = -nd}} R_\xi$$

into weight subspaces for the central torus $T \subset G_0$. Then, by Proposition 1, Theorem 1 is equivalent to Theorem 2 below.

Theorem 2. *The minimal degree d such that $R_d \neq 0$ is given by the first row of the following table, while the second row gives the dimensions of its T -homogeneous summands: (entries marked with an asterisk are conjectural).*

type	A.	B.	D.	E ₆	E ₇	E ₈	F ₄	G ₂
	1	4	2	2	2	2	4	3
	{1, 1}	{unknown}	{1*}	{1}	{1}	{1}	{1}	{1}

We also extract additional information about R in types A and G.

Proposition 2.

- (1) In type A the ring of invariants R is generated by a pair of elements of degree 1 with distinct T -weights.
- (2) In type G_2 the ring of invariants R is generated by a single element of degree 3.

We thus proceed to prove Theorem 2 and Proposition 2, first outlining the general strategy.

4.2. Strategy. The approach will differ depending on the degree as stated in Theorem 2 and on the Cartan type of \mathfrak{g} . The outliers, A_ℓ and G_2 , will be treated separately at the very beginning (these are the easy ones). Now, the remaining types are organised according to two binary criteria:

- (1) degree: quadric (D_ℓ, E_6, E_7, E_8) or quartic (B_ℓ, F_4), and
- (2) type: classical (B_ℓ, D_ℓ) or exceptional (E_6, E_7, E_8, F_4).

For classical algebras, we give a constructive proof involving an explicit invariant in R_d , $d = 2$ or 4 . For the exceptional ones, we argue non-constructively by computing the dimension of R_d using representation-theoretic methods. In type F_4 we use the standard way of branching a representation of \mathfrak{sp}_n to $\mathfrak{g}_0^{\text{ss}}$ in terms of formal characters. Since the complexity of this approach is roughly controlled by the size of the Weyl group of \mathfrak{g} , it is practically impossible to apply to E_8 (with its Weyl group of size nearly 7 million, compared to 1152 for F_4). Fortunately, a more refined method may be applied to find *quadric* invariants, involving only the quotient of the Weyl group of \mathfrak{g} by that of \mathfrak{g}_0 (for E_8 there are only 240 cosets).

In any case, it is not difficult to construct *candidates* for a nontrivial element of R_d . Indeed, due to the isomorphism (20), to give a nontrivial element of R_d is the same as to give a G_0^{ss} -invariant in $S^d \Lambda_0^n \mathfrak{g}_{-1}^*$ that is not contained in the ideal I (i.e., does not vanish on the Plücker-embedded $\text{LGr } \mathfrak{g}_{-1} \subset \mathbb{P} \Lambda_0^n \mathfrak{g}_{-1}$). We will now describe a way to produce G_0^{ss} -invariants in $S^d \Lambda_0^n \mathfrak{g}_{-1}^*$. In the case of classical algebras we will be able to show explicitly that they do not belong to I .

In the *quadric* case we are dealing with algebras of type D and E. It is an important observation that in all these cases n is *even*. As a consequence, the wedge product map $\Lambda_0^n \mathfrak{g}_{-1} \otimes \Lambda_0^n \mathfrak{g}_{-1} \rightarrow \det \mathfrak{g}_{-1} \simeq \mathbb{C}$ defines an $\mathfrak{sp}(\mathfrak{g}_{-1})$ -invariant quadratic form $b \in S^2 \mathfrak{g}_{-1}^*$. Of course, b vanishes on $\text{LGr } \mathfrak{g}_{-1}$ (the symplectic group acts transitively on the Lagrangian Grassmannian, see Subsection 3.3). Now, $\Lambda_0^n \mathfrak{g}_{-1}$ decomposes under $\mathfrak{g}_0^{\text{ss}}$ into a direct sum of invariant subspaces: for each of these the restriction of b is either non-degenerate or zero. The isotropic summands come in dual pairs, so that adding them produces a G_0^{ss} -invariant *orthogonal* decomposition of $\Lambda_0^n \mathfrak{g}_{-1}$ (we will see that the Plücker space is not a sum of two irreducible isotropic subspaces). Now, the restriction of b to each orthogonal summand may be again extended trivially to all of $\Lambda_0^n \mathfrak{g}_{-1}$ producing a G_0^{ss} -invariant quadric. Their sum yields b and thus vanishes on $\text{LGr } \mathfrak{g}_{-1}$, but one may expect that *not all* such restrictions vanish on their own.

In the *quartic* case we may exploit a universal construction. It is well known (see, for instance, [14]) that the G_0^{ss} -action on \mathfrak{g}_{-1} has quartic invariant, which may be defined for $x \in \mathfrak{g}_{-1}$ as $q(x) = \text{ad}_x^4 \in \text{Hom}(\mathfrak{g}_2, \mathfrak{g}_{-2}) \simeq \mathbb{C}$. Now, a G_0^{ss} -invariant quartic on $\Lambda_0^n \mathfrak{g}_{-1}$ is given by q^n under the natural map

$$S^n S^4 \mathfrak{g}_{-1}^* \rightarrow S^4 \Lambda^n \mathfrak{g}_{-1}^*.$$

In fact, we will show that q^n does not vanish on $\text{LGr } \mathfrak{g}_{-1}$ for both B_ℓ (where it is the lowest degree invariant) and D_ℓ (where it defines a quartic invariant *independent* from the square of the quadric).

4.3. Proof of Theorem 1 in type A. In this section we work out step-by-step the case A_{n+1} , i.e., when $G = \text{PGL}_{n+2}$, which is perhaps the simplest one. This ‘toy model’ will help the reader to better understand how the program sketched in the introductory Subsection 1.4 can be applied in practice. This is also the only case when the central torus T has rank two, which is another good reason to develop it in details.

Let PGL_{n+2} act naturally on the projective space \mathbb{P}^{n+1} .

Lemma 15. *The $(2n+1)$ -dimensional contact manifold $\mathbb{P}T^*\mathbb{P}^{n+1}$ is precisely the adjoint contact variety X of PGL_{n+2} .*

Proof. A point $L \in \mathbb{P}^{n+1}$ is a line in \mathbb{C}^{n+2} , and an element $H \in \mathbb{P}T^*\mathbb{P}^{n+1}$ is a tangent hyperplane to \mathbb{P}^{n+1} at L . As such, H is an hyperplane in \mathbb{C}^{n+2} containing the line L . In other words, $\mathbb{P}T^*\mathbb{P}^{n+1}$ can be identified with the space

$$\{(L, H) \mid L \subset H\} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{n+1*}$$

of $(1, n+1)$ -flags in \mathbb{C}^{n+2} .

Fix standard coordinates e_1, \dots, e_{n+2} on \mathbb{C}^{n+2} , together with their duals ϵ^i , and set the point

$$o := (L_0 = \langle e_1 \rangle, H_0 = \langle e_1, \dots, e_{n+1} \rangle = \ker \epsilon^{n+2})$$

as the origin of $\mathbb{P}T^*\mathbb{P}^{n+1}$. The Lie algebra of the stabiliser of o is

$$\mathfrak{p} = \left\{ \begin{pmatrix} \lambda & v & \mu \\ 0 & A & w \\ 0 & 0 & \nu \end{pmatrix} \mid A \in \mathfrak{gl}_n, v, w \in \mathbb{C}^n, \lambda + \nu + \text{tr } A = 0 \right\}.$$

Since $\dim \mathfrak{sl}_{n+2} - \dim \mathfrak{p} = ((n+2)^2 - 1) - (n^2 + 2n + 2) = 2n + 1$, the PGL_{n+2} -orbit through o is open in $\mathbb{P}T^*\mathbb{P}^{n+1}$.

Now we show that the map

$$\begin{aligned} \mathbb{P}T^*\mathbb{P}^{n+1} &\xrightarrow{F} \mathbb{P}(\mathfrak{sl}_{n+2}) \subset \mathbb{P}(\mathbb{C}^{n+2} \otimes \mathbb{C}^{n+2*}), \\ (L = \langle v \rangle, H = \ker \varphi) &\longmapsto [v \otimes \varphi] \end{aligned}$$

is well-defined, injective, PGL_{n+2} -equivariant, and it defines a contactomorphism on its image, which is precisely X .

The class $[v \otimes \varphi]$ is well-defined because both v and φ are defined up to a projective factor, and $v \otimes \varphi$ indeed belongs to \mathfrak{sl}_{n+2} , since from $L \subset H$ it follows that $\text{tr}(v \otimes \varphi) = \varphi(v) = 0$.

By construction,

$$F(o) = [e_1 \otimes \epsilon^{n+2}],$$

where $e_1 \otimes \epsilon^{n+2}$ is the highest weight vector of \mathfrak{sl}_{n+2} , whence

$$\text{PGL}_{n+2} \cdot [e_1 \otimes \epsilon^{n+2}] = X,$$

by the very definition of adjoint variety (see Definition 1).

The PGL_{n+2} -equivariance of F is obvious, since

$$g \cdot (L, H) = (g(L), g(H)) = (\langle g(v) \rangle, \ker g^*(\varphi)) \mapsto [g(v) \otimes g^*(\varphi)] = g \cdot [v \otimes \varphi], \quad \forall g \in \text{PGL}_{n+2}.$$

Moreover, F is injective being the restriction of the Segre embedding $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1*} \subset \mathbb{P}(\mathbb{C}^{n+2} \otimes \mathbb{C}^{n+2*})$, and its image coincides with X . Indeed, if $[h] \in \mathbb{P}(\mathfrak{sl}_{n+2})$, where h is a rank-one homomorphism, then $(\text{im } h, \ker h) \in \mathbb{P}T^*\mathbb{P}^{n+1}$ and

$$[h] = F((\text{im } h, \ker h)).$$

It remains to prove that F realises a contactomorphism between the contact structures on $\mathbb{P}T^*\mathbb{P}^{n+1}$ and X . By homogeneity, we can simply show that $T_o F$ sends the contact hyperplane

$$H_0 \oplus T_{H_0}(\mathbb{P}T_{L_0}^*\mathbb{P}^{n+1}) \subset T_{L_0}\mathbb{P}^{n+1} \oplus T_{H_0}(\mathbb{P}T_{L_0}^*\mathbb{P}^{n+1}) = T_o(\mathbb{P}T^*\mathbb{P}^{n+1})$$

to the contact hyperplane of $T_{F(o)}X$. The latter is better described in terms of the cone \widehat{X} over X , inside \mathfrak{sl}_{n+2} : it is the subspace

$$\ker \text{ad}_{e_1 \otimes \epsilon^{n+2}} \subset [\mathfrak{sl}_{n+2}, e_1 \otimes \epsilon^{n+2}] = T_{e_1 \otimes \epsilon^{n+2}} \widehat{X}.$$

A curve $\gamma(t) := (\langle v_t \rangle, \ker \phi_t)$ belongs to the contact plane at o if and only if $\epsilon^{n+2}(v'_0) = 0$, that is, if the horizontal projection of $\gamma(t)$ keeps, to first order, the line $\langle v_t \rangle$ inside the hyperplane $\ker \phi_t$. Observe that, since ϕ'_0 is the velocity of a curve of hyperplanes containing L_0 , we have also $\phi'_0(e_1) = 0$. Consider now the curve

$$(21) \quad t \mapsto e_1 \otimes \epsilon^{n+2} + (v'_0 \otimes \epsilon^{n+2} + e_1 \otimes \phi'_0)t + o(t^2)$$

in \mathfrak{sl}_{n+2} , whose projectivisation is precisely $F_*\gamma$. Since

$$\text{ad}_{e_1 \otimes \epsilon^{n+2}}(v'_0 \otimes \epsilon^{n+2} + e_1 \otimes \phi'_0) = (\epsilon^{n+2}(v'_0) + \phi'_0(e_1))e_1 \otimes \epsilon^{n+2} = 0,$$

the velocity at 0 of (21) belongs to the contact hyperplane of \widehat{X} at $e_1 \otimes \epsilon^{n+2}$, whence the velocity at 0 of $F_*\gamma$ at $F(o)$ belongs to the contact hyperplane of X at $F(o)$. \square

Recall (see Subsection 3.2) the notion of contact grading.

Corollary 3. *The contact grading (4) of \mathfrak{sl}_{n+2} is*

$$(22) \quad \mathfrak{sl}_{n+2} = \underbrace{\mathbb{C}}_{\mathfrak{g}_{-2}} \oplus \underbrace{(\mathbb{C}^n \oplus \mathbb{C}^{n*})}_{\mathfrak{g}_{-1}} \oplus \underbrace{(\mathfrak{sl}_n \oplus \mathbb{C}^2)}_{\mathfrak{g}_0} \oplus \underbrace{(\mathbb{C}^{n*} \oplus \mathbb{C}^n)}_{\mathfrak{g}_1} \oplus \underbrace{\mathbb{C}^*}_{\mathfrak{g}_2}.$$

Proof. The stabiliser of o is

$$\mathfrak{p} = \left\{ \begin{pmatrix} \lambda & v & \mu \\ 0 & A & w \\ 0 & 0 & \nu \end{pmatrix} \mid A \in \mathfrak{gl}_n, v, w \in \mathbb{C}^n, \lambda + \nu + \text{tr } A = 0 \right\}.$$

Then

$$\begin{pmatrix} \lambda & v & \mu \\ 0 & A & w \\ 0 & 0 & \nu \end{pmatrix} \mapsto \left(\underbrace{A - \frac{1}{n} \text{tr } A \text{Id}_n, \text{tr } A, \lambda + \nu}_{\mathfrak{g}_0}, \underbrace{\begin{pmatrix} 0 & v & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix}}_{\mathfrak{g}_1}, \underbrace{\mu}_{\mathfrak{g}_2} \right)$$

defines an isomorphism $\mathfrak{p} \cong \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. \square

Corollary 4. *The twisted symplectic form on \mathfrak{g}_{-1} is the unique one extending the standard pairing between \mathbb{C}^n and \mathbb{C}^{n*} .*

Proof. Just observe that

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix} \right] = -v \cdot w$$

is the matrix counterpart of the pairing

$$\mathbb{C}^n \times \mathbb{C}^{n*} \ni (x, \xi) \mapsto \xi(x) \in \mathbb{C}$$

\square

Let us stress here that the identification of the particular summands in (22) is a representation of the semi-simple part $G_0^{\text{ss}} \simeq \text{SL}_n$. Formula (10) reads now $G_0 = \text{SL}_n \times T$, with a central torus T of rank 2. As a next step, we describe its action on \mathfrak{g}_{-1} (this determines the action on all the remaining graded components of \mathfrak{g}).

Lemma 16. *There is an torus isomorphism $T \simeq (\mathbb{C}^\times)^2$ such that $(a, b) \in (\mathbb{C}^\times)^2$ sends an element (x, ξ) of the twisted symplectic space $\mathbb{C}^n \oplus \mathbb{C}^{n*}$ to $(a^{-1}x, a^{-1}b^{-1}\xi)$.*

In particular, the first factor $\mathbb{C}^\times \subset T$ acts on \mathfrak{g}_{-1}^* by the usual scalar multiplication. Thus, it recovers the standard grading on $S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^*$, up to a rescaling by n .

Proof. Let $\nu_1, \nu_2 \in \hat{T}$ be the characters with which T acts on, respectively, \mathbb{C}^n and \mathbb{C}^{n*} . We want to show that there is a torus isomorphism $T \simeq (\mathbb{C}^\times)^2$ inducing $\hat{T} \simeq \mathbb{Z}^2$ such that ν_1, ν_2 become, respectively, $(-1, 0)$ and $(-1, -1)$. Since the determinant of these two vectors is 1, this is equivalent to showing that ν_1, ν_2 generate the character lattice \hat{T} .

It is convenient to consider first the simply-connected cover $\mathrm{SL}_{n+2} \rightarrow \mathrm{PGL}_{n+2}$ and the preimage $\tilde{G}_0 \subset \mathrm{SL}_{n+2}$ of $G_0 \subset \mathrm{PGL}_{n+2}$. We have $\mathrm{PGL}_{n+2} = \mathrm{SL}_{n+2} / \mu_{n+2}$ where $\mu_{n+2} \subset \mathrm{SL}_{n+2}$ is the group of $n+2$ nd roots of unity times the identity matrix. Furthermore, \tilde{G}_0 decomposes as $\tilde{G}_0 = \mathrm{SL}_n \times \tilde{T}$, where $\mathrm{SL}_n \subset \mathrm{SL}_{n+2}$ is the obvious embedding induced by the block decomposition (22) considered above, while \tilde{T} is a central torus containing μ_{n+2} . Now, the projection restricts to $\tilde{G}_0 \simeq \mathrm{SL}_n \times \tilde{T} \rightarrow \mathrm{SL}_n \times T \simeq G_0$, induced by the identity on SL_n and by the projection $\tilde{T} \rightarrow T \simeq T / \mu_{n+2}$. Dualising the latter, we have an inclusion of \hat{T} as a sub-lattice of index $n+2$ in the character lattice of \tilde{T} .

Identifying \tilde{T} with $(\mathbb{C}^\times)^2$ by the embedding

$$(\mathbb{C}^\times)^2 \ni (a, b) \mapsto \begin{pmatrix} a & & \\ & b & \\ & & a^{-1}b^{-n} \end{pmatrix} \in \tilde{G}_0$$

we obtain an identification of the character lattice of \tilde{T} with \mathbb{Z}^2 . Using the adjoint action of the above matrix on a matrix corresponding to $(x, \xi) \in \mathfrak{g}_{-1}$, it is straightforward to compute the characters of \mathbb{C}^n and \mathbb{C}^{n*} : these are, respectively, $\nu_1 = (-1, 1)$ and $\nu_2 = (-1, -n-1)$. Computing the determinant of these two elements we find that ν_1, ν_2 generate a sub-lattice of index $n+2$ in the character lattice of \tilde{T} . On the other hand, they are contained in \hat{T} , a sub-lattice of the same index. Hence ν_1, ν_2 generate \hat{T} . \square

Let us now decompose the dual Plücker space $\Lambda_0^n \mathfrak{g}_{-1}^*$ into G_0^{ss} -irreducible subspaces and find the characters with which T acts on the summands (see Subsection 3.4). Since G_0 preserves the Lagrangian decomposition $\mathfrak{g}_{-1} = \mathbb{C}^n \oplus \mathbb{C}^{n*}$, the decomposition of the Plücker space (18) takes a particularly simple form:

$$\begin{aligned} \Lambda_0^n \mathfrak{g}_{-1}^* &= \Lambda_0^n (\mathbb{C}^{n*} \oplus \mathbb{C}^n) \\ (23) \quad &= \bigoplus_{i=0}^n S_0^2(\Lambda^i \mathbb{C}^{n*}) \subset \bigoplus_{i=0}^n (\Lambda^i \mathbb{C}^{n*})^{\otimes 2} = \bigoplus_{i=0}^n (\Lambda^i \mathbb{C}^{n*}) \otimes (\Lambda^{n-1-i} \mathbb{C}^n), \end{aligned}$$

$$(24) \quad \Lambda_0^n \mathfrak{g}_{-1}^* = \mathbb{C} \oplus S_0^2 \mathbb{C}^{n*} \oplus \cdots \oplus S_0^2(\Lambda^{n-1} \mathbb{C}^{n*}) \oplus S_0^2(\Lambda^n \mathbb{C}^{n*}).$$

The step (23) may use some extra comment. First, we decomposed n -forms on $\mathbb{C}^{n*} \oplus \mathbb{C}^n$ as products of forms on each summand, and then we used Poincaré duality. Finally, one checks that a bilinear form on $\Lambda^i \mathbb{C}^n$ belongs to the kernel of (7) if and only if it is symmetric and trace-free. So, (24) represents the decomposition of the space of linear functions on the Plücker embedding space of $X_o^{(1)}$ into G_0^{ss} -irreducible submodules. The bidegree of the summand identified with $S_0^2 \Lambda^i \mathbb{C}^{n*}$ is computed using Lemma 16 as (n, i) . Clearly, there are only two one-dimensional summands in (24): the first and the last. We thus have

$$\mathbb{C} = R_{(n,0)}, \quad S^2(\Lambda^n \mathbb{C}^{n*}) = R_{(n,n)}.$$

We conclude that $\mathrm{Inv}(X, \mathbf{A}_{n+1})$ contains exactly two elements of degree 1, i.e., the lowest-degree invariants we were looking for.⁵

For completeness, we state the following result.

Lemma 17. *$R_{(n,0)}$ and $R_{(n,n)}$ generate R .*

Proof. We work geometrically on the Lagrangian Grassmannian $\mathrm{LGr} \mathfrak{g}_{-1} \subset \mathbb{P} \Lambda_0^n \mathfrak{g}_{-1}$. As above, we use a bi-lagrangian splitting $\mathfrak{g}_{-1} = \mathbb{C}^n \oplus \mathbb{C}^{n*}$ as in Lemmas 5 and 8. Recall that we have dense open subsets $U \simeq S^2 \mathbb{C}^{n*}$ and $U' \simeq S^2 \mathbb{C}^n$ in $\mathrm{LGr} \mathfrak{g}_{-1}$, consisting of Lagrangian subspaces $L \subset \mathfrak{g}_{-1}$ with non-degenerate projections onto \mathbb{C}^n , respectively \mathbb{C}^{n*} . Let $H = \mathrm{LGr} \mathfrak{g}_{-1} \setminus U$ and $H' = \mathrm{LGr} \mathfrak{g}_{-1} \setminus U'$. All these subsets are G_0^{ss} -invariant, and the action of the latter on U and U' coincides with its natural linear action on $S^2 \mathbb{C}^{n*}$ and $S^2 \mathbb{C}^n$. We may view $H' \cap U$ as the locus of those symmetric maps $\mathbb{C}^n \rightarrow \mathbb{C}^{n*}$ that are not invertible, and analogously for $H \subset U'$. It follows that $H' \cap U$ is the hypersurface in $S^2 \mathbb{C}^{n*}$ cut out by the determinant $\det : S^2 \mathbb{C}^{n*} \rightarrow (\det \mathbb{C}^{n*})^2 \simeq \mathbb{C}$. Thus the hyperplane section H' is defined by an element of $R_{(n,0)} \subset \Lambda_0^n \mathfrak{g}_{-1}^*$ and, dually, H is defined by an element of $R_{(n,n)}$.

Suppose now $r \in R_d$ is a nonzero invariant and let D be the associated hypersurface (recall that to us this means an effective divisor) on $\mathrm{LGr} \mathfrak{g}_{-1}$. We may write $D = aH + \sum b_i Z_i$ where the Z_i are codimension one subvarieties of $\mathrm{LGr} \mathfrak{g}_{-1}$ such that $Z_i = \overline{Z_i \cap U}$. Since $D_U = \sum b_i Z_i \cap U$ is necessarily G_0^{ss} -invariant, it will be enough to check that the ring of SL_n -invariants in $\mathbb{C}[S^2 \mathbb{C}^{n*}]$ is generated by \det , for then it follows that D_U is a multiple of $H' \cap U$. That is a classical result (see, e.g., [20]). \square

⁵ We stress that in (24) the submodules $S_0^2 \mathbb{C}^{n*}$ and $S_0^2(\Lambda^{n-1} \mathbb{C}^{n*})$ coincide in fact with $S^2 \mathbb{C}^{n*}$ and $S^2(\Lambda^{n-1} \mathbb{C}^{n*})$, respectively, since the entries of a symmetric matrix, as well as of its cofactor matrix are independent. Only for $n \geq 4$ the trace-free submodules can be proper ones. See, e.g., the discussion [16], where the whole decomposition (24) is obtained.

4.4. Proof of Theorem 1 in type G. We move now to \mathfrak{g} of type G, that is, to the case when G is the 14-dimensional Lie group G_2 . Being the smallest amongst the exceptional Lie groups, G_2 is perhaps the best understood one, and this section does not add anything new (see [1] for a thorough review of G_2 , and also [21]).

The contact grading (4) reads now

$$\mathrm{Lie}(G_2) = \underbrace{\mathbb{C}}_{\mathfrak{g}_{-2}} \oplus \underbrace{S^3\mathbb{C}^2}_{\mathfrak{g}_{-1}} \oplus \underbrace{(\mathfrak{sl}_2 \oplus \mathbb{C})}_{\mathfrak{g}_0} \oplus \underbrace{S^3\mathbb{C}^{2*}}_{\mathfrak{g}_1} \oplus \underbrace{\mathbb{C}^*}_{\mathfrak{g}_2},$$

where \mathfrak{g}_{-1} is the 4-dimensional irreducible representation of $G_0^{\mathrm{ss}} \simeq \mathrm{SL}_2$.

Observe that $X_o^{(1)}$ is the 3-dimensional Grassmannian of Lagrangian 2-planes in a 4-dimensional symplectic space, which is a quadric in \mathbb{P}^4 (see, e.g., [24]). Accordingly, $\Lambda_0^2 \mathfrak{g}_{-1}$ must be an irreducible 5-dimensional SL_2 -module, i.e., $\Lambda_0^2 \mathfrak{g}_{-1} \cong S^4 \mathbb{C}^2$, and (11) becomes

$$R = (S^\bullet(S^4 \mathbb{C}^{2*})/I)^{\mathrm{SL}_2} = \mathbb{C}[S^4 \mathbb{C}^2]^{\mathrm{SL}_2}/I^{\mathrm{SL}_2}.$$

It is a classical result that

$$\mathbb{C}[S^4 \mathbb{C}^2]^{\mathrm{SL}_2} = \mathbb{C}[q, c],$$

where q is a quadric and c a cubic (see, e.g., [20]). Since $I = \langle q \rangle$, the lowest-degree constituent of $\mathrm{Inv}(X, G_2)$ is $\mathbb{P}R_3 = \{[c]\}$. Observe that the grading of R induced by the central torus in G_0 coincides with the standard grading.

4.5. Proof of Theorem 1 in types B and D. In this section we deal with both the cases when \mathfrak{g} is of type B and D, since they both correspond to the special orthogonal Lie group. We adopt a common approach, by stressing the differences due to the parity of n , where $2n$ is the dimension of \mathfrak{g}_0 (cf. (5)).

Accordingly, the contact grading (4) becomes

$$\mathfrak{so}_{n+4} = \underbrace{\mathbb{C}}_{\mathfrak{g}_{-2}} \oplus \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^n}_{\mathfrak{g}_{-1}} \oplus \underbrace{(\mathfrak{gl}_2 \oplus \mathfrak{so}_n)}_{\mathfrak{g}_0} \oplus \underbrace{(\mathbb{C}^2 \otimes \mathbb{C}^n)^*}_{\mathfrak{g}_1} \oplus \underbrace{\mathbb{C}^*}_{\mathfrak{g}_2}.$$

Since $G_0^{\mathrm{ss}} \simeq \mathrm{SL}_2 \times \mathrm{SO}_n$, the decomposition (18) of the Plücker embedding space into $\mathfrak{g}_0^{\mathrm{ss}}$ -irreducible submodules reads

$$(25) \quad \Lambda_0^n \mathfrak{g}_{-1} \simeq \begin{cases} \bigoplus_{a+b=\frac{n-1}{2}} S^{2a+1} \mathbb{C}^2 \otimes S_0^2 \Lambda^b \mathbb{C}^n & n \text{ odd,} \\ \bigoplus_{a+b=\frac{n}{2}} S^{2a} \mathbb{C}^2 \otimes S_0^2 \Lambda^b \mathbb{C}^n & n \text{ even,} \end{cases}$$

where $S_0^2 \Lambda^b \mathbb{C}^n$ is: an irreducible representation of SO_n with highest weight being twice the highest weight of $\Lambda^b \mathbb{C}^n$ for $b < n/2$; or the direct sum of the irreducible representations whose highest weights are twice the highest weights of the two summands of $\Lambda^{n/2} \mathbb{C}^n$ for $b = n/2$. This follows from the decomposition of the $(\mathrm{SL}_2 \times \mathrm{SL}_n)$ -representation

$$\Lambda^n(\mathbb{C}^2 \otimes \mathbb{C}^n) = \bigoplus_{|\lambda|=n} \Sigma_\lambda(\mathbb{C}^2) \otimes \Sigma_{\lambda^*}(\mathbb{C}^n)$$

where the sum is over Young diagrams of size n , λ^* denotes the transpose of λ , and Σ_λ is the Schur functor associated with λ (see [10, Exercise 6.11 b]). Indeed, one sees that the only diagrams entering the sum are $\alpha = [n-b, b]$ in row-length notation, $0 \leq b \leq n/2$, with $\Sigma_{[n-b, b]}(\mathbb{C}^2) \simeq S^{n-2b} \mathbb{C}^2$ and $\Sigma_{[n-b, b]^*}(\mathbb{C}^n) \simeq (\mathrm{End} \Lambda^b \mathbb{C}^n)_0$. The latter denotes the unique irreducible SL_n -subrepresentation in $\mathrm{End} \Lambda^b \mathbb{C}^n$ containing the image of SL_n under the representation map $\mathrm{SL}_n \rightarrow \mathrm{GL}(\Lambda^b \mathbb{C}^n)$. Then, reducing to $\mathrm{SL}_2 \times \mathrm{SO}_n \subset \mathrm{SL}_2 \times \mathrm{SL}_n$ and taking a quotient by $\omega \wedge \Lambda^{n-2}(\mathbb{C}^2 \otimes \mathbb{C}^n)$, one arrives at (25). As an immediate consequence of (25) we have $\mathbb{P}R_1 = \emptyset$, that is, there are no G_0^{ss} -invariant hyperplanes.

Remark 4. Observe that the Plücker embedding space $\Lambda_0^n \mathfrak{g}_{-1}$ is equipped with a nondegenerate pairing

$$\begin{aligned} \Lambda_0^n \mathfrak{g}_{-1} \times \Lambda_0^n \mathfrak{g}_{-1} &\longrightarrow \Lambda^{2n} \mathfrak{g}_{-1} \simeq \mathbb{C} \\ (\phi, \psi) &\longmapsto \phi \wedge \psi, \end{aligned}$$

which is a quadratic form for n even (i.e., \mathfrak{g} of type D), and a symplectic form for n odd (i.e., \mathfrak{g} of type B). In the even case the corresponding null quadric contains $X_o^{(1)}$, and thus does not produce a nontrivial invariant of degree 2. However, we may use the restriction of the corresponding bilinear form to a G_0 -invariant subspace of the Plücker space.

Proposition 3. *For \mathfrak{g} of type D, there is a nontrivial element $[B]$ in $\mathbb{P}R_2$.*

Proof. Observe that the map

$$\begin{aligned} \Lambda_0^n \mathfrak{g}_{-1} &\xrightarrow{\pi} S^n \mathbb{C}^2 \otimes \Lambda^n \mathbb{C}^n \equiv S^n \mathbb{C}^2 \\ \xi_1 \otimes v_1 \wedge \cdots \wedge \xi_n \otimes v_n &\longmapsto (\xi_1 \odot \cdots \odot \xi_n) \otimes (v_1 \wedge \cdots \wedge v_n) \end{aligned}$$

is surjective and G_0^{ss} -equivariant, and that the SL_2 -invariant projection

$$S^2 S^n \mathbb{C}^2 \xrightarrow{q} S^n \Lambda^2 \mathbb{C}^2 \simeq \mathbb{C}$$

defines an SL_2 -invariant quadratic form q on $S^n \mathbb{C}^{2*}$. Therefore, $B := \pi^*(q)$ is a quadratic form on $\Lambda_0^n \mathfrak{g}_{-1}$.

It remains to show that B does belong to the ideal I (cf. (17)), that is, that B does not vanish on $X_o^{(1)}$. To this end, we to show that $B(\phi) \neq 0$ for some $\phi \in \Lambda_0^n \mathfrak{g}_{-1}$ such that $[\phi] \in X_o^{(1)}$. Fix an orthonormal basis e_1, \dots, e_n of \mathbb{C}^n . Let $\eta \in S^n \mathbb{C}^2$ be such that $q(\eta) \neq 0$, and fix a factorisation $\eta = \xi_1 \cdots \xi_n$ with $\xi_i \in \mathbb{C}^2$, $1 \leq i \leq n$. Now, consider the

linear subspace $L = \langle \xi_1 \otimes e_1, \dots, \xi_n \otimes e_n \rangle \subset \mathfrak{g}_{-1}$. It is by construction Lagrangian, and furthermore its representing n -form $\phi = (\xi_1 \otimes e_1) \wedge \dots \wedge (\xi_n \otimes e_n)$ satisfies $B(\phi) = q(\eta) \neq 0$. \square

This proves the part of Theorem 2 referring to type D. In type B, Proposition 4 below shows that no such invariant quadric exists.

Proposition 4. *For \mathfrak{g} of type B, we have $\mathbb{P}R_2 = \emptyset$.*

Proof. From the decomposition formula (25) of the Plücker embedding space $\Lambda_0^n \mathfrak{g}_{-1}$ it follows that there are no non-trivial G_0^{ss} -invariants in $S^2(\Lambda_0^n \mathfrak{g}_{-1})^*$. Indeed, each summand is the tensor product of a symplectic module and an orthogonal module, hence symplectic. Furthermore, no two summands are mutually dual. \square

The next step in type B is to look for cubic invariants. Along the same line as Proposition 4, we have:

Proposition 5. *For \mathfrak{g} of type B, we have $\mathbb{P}R_3 = \emptyset$.*

Proof. Identifying integral weights of SL_2 with \mathbb{Z} , and letting $S_a = S^a \mathbb{C}^2$, we have that the set of weights of $S_{a_1} \otimes \dots \otimes S_{a_r}$ is contained in $a_1 + \dots + a_r + 2\mathbb{Z}$. In particular, the tensor product of an odd number of even-dimensional representations of SL_2 cannot contain the trivial representation. Since a G_0^{ss} -invariant in a tensor power of $\Lambda_0^n \mathfrak{g}_{-1}^*$ necessarily decomposes into summands which are products of an SL_2 -invariant and an SO_n -invariant, it follows that there are no odd-degree G_0^{ss} -invariants on $\Lambda_0^n \mathfrak{g}_{-1}$ whatsoever. \square

The final result of this section is common to both types D and B. In type B it provides the sought-for lowest-degree invariant, thus proving the corresponding part of Theorem 2, whereas in type D it exhibits another interesting element of $\text{Inv}(X, G)$.

We begin by observing that, by the classical invariant theory of SL_2 and SO_n , there is a unique one-dimensional subspace in the G_0^{ss} -irreducible decomposition of $S^4 \mathfrak{g}_{-1}^*$, given dually by the projection

$$S^4(\mathbb{C}^2 \otimes \mathbb{C}^n) \rightarrow S^2 \Lambda^2 \mathbb{C}^2 \otimes \ker \left[S^2 \Lambda^2 \mathbb{C}^n \xrightarrow{\wedge} \Lambda^4 \mathbb{C}^n \right] \xrightarrow{\text{id} \otimes \langle \cdot, \cdot \rangle} (\det \mathbb{C}^2)^2 \otimes \mathbb{C} \simeq \mathbb{C}.$$

The corresponding quartic $q : S^4(\mathbb{C}^2 \otimes \mathbb{C}^n) \rightarrow \mathbb{C}$ is defined as

$$q(\xi_1 \otimes e_1, \xi_2 \otimes e_2, \xi_3 \otimes e_3, \xi_4 \otimes e_4) = \epsilon(\xi_1, \xi_2) \epsilon(\xi_3, \xi_4) [\langle e_1, e_3 \rangle \langle e_2, e_4 \rangle - \langle e_2, e_3 \rangle \langle e_1, e_4 \rangle]$$

where $\epsilon \in \Lambda^2 \mathbb{C}^{2*}$ is a volume form and $\langle \cdot, \cdot \rangle$ the SO_n -invariant inner product on $\Lambda^2 \mathbb{C}^n$. It is necessarily proportional to the ‘canonical’ G_0^{ss} -invariant quartic described in Subsection 4.2.

Proposition 6. *For \mathfrak{g} of type B or D, we have that $[q^n] \in \mathbb{P}R_4$ is well-defined.*

Proof. Consider as before a Lagrangian subspace of the form $L = \langle \xi_1 \otimes e_1, \dots, \xi_n \otimes e_n \rangle \subset \mathfrak{g}_{-1}$, where e_1, \dots, e_n is an orthonormal basis in \mathbb{C}^n , while $\xi_1, \dots, \xi_n \in \mathbb{C}^2$ is a general n -tuple. We compute $q^n(\phi)$ for $\phi = \bigwedge_{i=1}^n (\xi_i \otimes e_i)$:

$$\begin{aligned} q^n(\phi) &= \sum_{\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}} \text{sgn}(\sigma^{(1)} \sigma^{(2)} \sigma^{(3)}) \\ &\quad \times \prod_{i=1}^n \epsilon(\xi_i, \xi_{\sigma^{(1)} i}) \epsilon(\xi_{\sigma^{(2)} i}, \xi_{\sigma^{(3)} i}) \\ &\quad [\langle e_i, e_{\sigma^{(2)} i} \rangle \langle e_{\sigma^{(1)} i}, e_{\sigma^{(3)} i} \rangle - \langle e_i, e_{\sigma^{(3)} i} \rangle \langle e_{\sigma^{(1)} i}, e_{\sigma^{(2)} i} \rangle] \\ &= \sum_{\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}} \text{sgn}(\sigma^{(1)} \sigma^{(2)} \sigma^{(3)}) \prod_{i=1}^n \left[\delta_{\sigma^{(2)} i}^i \delta_{\sigma^{(3)} i}^{\sigma^{(1)} i} - \delta_{\sigma^{(3)} i}^i \delta_{\sigma^{(2)} i}^{\sigma^{(1)} i} \right] \\ &\quad \times \prod_{i=1}^n \epsilon(\xi_i, \xi_{\sigma^{(1)} i}) \epsilon(\xi_{\sigma^{(2)} i}, \xi_{\sigma^{(3)} i}). \end{aligned}$$

Note that this way we have

$$q^n(\phi) = Q(\xi_1 \cdots \xi_n)$$

for certain SL_2 -invariant quartic Q on $S^n \mathbb{C}^2$. We only need to check that $Q \neq 0$. We further rewrite:

$$Q(\xi_1 \cdots \xi_n) = (-1)^n \sum_{\sigma} \left(\prod_i \epsilon(\xi_i, \xi_{\sigma i})^2 \right) \times \left(\text{sgn}(\sigma) \sum_{J \subset \{1, \dots, n\}} C_{\sigma, J} \right),$$

where

$$C_{\sigma, J} = \text{sgn}(\sigma^J) \text{sgn}(\sigma^{J^c}), \quad \sigma^J(i) = \begin{cases} i & i \in J, \\ \sigma(i) & i \notin J, \end{cases}$$

and $\text{sgn}(\sigma^J) = 0$ if σ^J is not a permutation. The sum over J may be restricted to σ -invariant sets, in which case $C_{\sigma, J} = \text{sgn}(\sigma)$ and we obtain

$$Q(\xi_1 \cdots \xi_n) = (-1)^n \sum_{\sigma} \left(\prod_i \epsilon(\xi_i, \xi_{\sigma i})^2 \right) \cdot \#\{J \subset \{1, \dots, n\} \mid \sigma J = J\}.$$

Choosing ξ_1, \dots, ξ_n in \mathbb{R}^n such that $\epsilon(\xi_i, \xi_j) = 0$ if and only if $i = j$, we find that $(-1)^n Q(\xi_1, \dots, \xi_n)$ is a sum of non-negative reals, with positive terms corresponding to fix-point free σ . Hence $Q(\xi_1, \dots, \xi_n) \neq 0$ and thus $q^n(\phi) \neq 0$. \square

4.6. Representation-theoretic setup. Having dealt with types A, B, D and G in a rather direct manner, we shall need to resort to more abstract methods in order to handle the remaining types E and F. This subsection introduces some representation-theoretic tools that are valid in greater generality,⁶ by picking up where we left off Subsection 3.2. We will use the language of modules over the universal enveloping $U(\mathfrak{g}_0)$ rather than representations of G_0 . Since the latter is connected, this does not change the notion of invariance.

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and sets $\Delta \subset \Phi^+ \subset \Phi$ of simple and positive roots within the root system of \mathfrak{g} with respect to $U(\mathfrak{h})$, compatible with the grading. In particular, \mathfrak{g}_2 is the root space of the longest positive root. Given a $U(\mathfrak{g})$ -module M , let $C^\bullet(\mathfrak{g}_-, M)$ denote the cochain complex computing the Lie algebra cohomology of \mathfrak{g}_- with values in M . If M is graded compatibly with \mathfrak{g} , let $C_i^\bullet(\mathfrak{g}_-, M)$ denote the homogeneous degree i subcomplex. Use $Z_i^\bullet, B_i^\bullet, H_i^\bullet$ to denote the spaces of cocycles, coboundaries and the cohomology, respectively.

Let us identify $\mathfrak{sp}(\mathfrak{g}_{-1})$ with \mathfrak{sp}_n together with a choice of a Cartan and Borel subalgebra. Let $\lambda_1, \dots, \lambda_n$ be the fundamental weights of \mathfrak{sp}_n and V_λ the simple $U(\mathfrak{sp}_n)$ -module of highest weight λ . Let W be the Weyl group of \mathfrak{g} , with $W^\mathfrak{p}$ the subset consisting of words $w \in W$ such that $w\rho$ is \mathfrak{g}_0 -dominant with ρ being the sum of all fundamental weights of \mathfrak{g} . Let $W_i^\mathfrak{p}$ denote the subset of $W^\mathfrak{p}$ consisting of words w of length i .

Lemma 18. *For each $1 \leq i \leq n$ we have $U(\mathfrak{g}_0)$ -module isomorphisms*

$$V_{\lambda_i} \simeq \Lambda_0^i \mathfrak{g}_{-1} \simeq H^i(\mathfrak{g}_-, \mathbb{C})^* \simeq \bigoplus_{w \in W_i^\mathfrak{p}} L(w \cdot 0)$$

where $L(\lambda)$ denotes the simple $U(\mathfrak{g}_0)$ -module with highest weight λ .

Proof. Note first that each $C^i(\mathfrak{g}_-, \mathbb{C})$ is the direct sum of subspaces of degrees i and $i+1$. Considering the part of the complex computing the degree i subspace $H_i^i(\mathfrak{g}_-, \mathbb{C})$, we have the following identifications:

$$\begin{array}{ccccc} C^{i-1}_i(\mathfrak{g}_-, \mathbb{C}) & \xrightarrow{\partial} & C^i_i(\mathfrak{g}_-, \mathbb{C}) & \xrightarrow{\partial} & C^{i+1}_i(\mathfrak{g}_-, \mathbb{C}) \\ \parallel & & \parallel & & \parallel \\ \mathfrak{g}_{-2}^* \otimes \Lambda^{i-2} \mathfrak{g}_{-1}^* & \xrightarrow{\omega \wedge} & \Lambda^i \mathfrak{g}_{-1}^* & \longrightarrow & 0, \end{array}$$

where ω is the twisted symplectic form (15). We thus have a $U(\mathfrak{g}_0)$ -module isomorphism

$$H_i^i(\mathfrak{g}_-, \mathbb{C}) \simeq \Lambda^i \mathfrak{g}_{-1}^* / (\omega \wedge (\mathfrak{g}_{-2}^* \otimes \Lambda^{i-2} \mathfrak{g}_{-1}^*))$$

and its dual

$$H_i^i(\mathfrak{g}_-, \mathbb{C})^* \simeq \Lambda_0^i \mathfrak{g}_{-1}.$$

On the other hand, along similar lines we get $H_{i+1}^i(\mathfrak{g}_-, \mathbb{C}) = 0$ by injectivity of the wedge map $\omega \wedge : \Lambda^i \mathfrak{g}_{-1}^* \rightarrow \Lambda^{i+2} \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}^*$ for $i < n$ (this is a standard fact on symplectic vector spaces). Hence $H^i(\mathfrak{g}_-, \mathbb{C}) = H_i^i(\mathfrak{g}_-, \mathbb{C})$ and thus

$$\Lambda_0^i \mathfrak{g}_{-1} \simeq H^i(\mathfrak{g}_-, \mathbb{C})^*.$$

The remaining two isomorphisms are given by Kostant's theorem (the decomposition of cohomology into simple modules [18]) and standard representation theory of \mathfrak{sp}_n (identification of fundamental modules with Lagrangian exterior powers). \square

4.7. Computing the space of quadric invariants for even n . We will now introduce a representation-theoretic method to compute the dimension of the space of invariant quadrics valid whenever n is even (i.e., when the Plücker space has an \mathfrak{Sp}_n -invariant symmetric bilinear form). That is, it can be applied in type E (which is our main point of interest here) as well as D.

We denote by I the ideal of $\text{LGr } \mathfrak{g}_{-1}$ in $S^\bullet \Lambda_0^n \mathfrak{g}_{-1}^*$. By Corollary 2, its degree 2 part I_2 may be identified with the complement of $V_{2\lambda_n}$ in $S^2 V_{\lambda_n}$. Accordingly, R_2 becomes identified with the space of $\mathfrak{g}_0^{\text{ss}}$ -invariants in the simple $U(\mathfrak{sp}_n)$ -module $V_{2\lambda_n}$. We should thus decompose the latter into simple $U(\mathfrak{g}_0)$ -submodules and look for rank-one summands. In order to use the Weyl-group description of Lemma 18, we need to express modules of the form $V_{2\lambda_i}$ in terms of tensor products of the fundamental modules.

Lemma 19. *Assume n even. Let λ_i denote the i^{th} fundamental weight of \mathfrak{sp}_n , and set for convenience $\lambda_0 = 0$. We then have:*

$$S^2 V_{\lambda_i} \simeq \sum_{0 \leq j \leq i/2} V_{2\lambda_{i-2j}} \oplus \sum_{\substack{i/2 - n \leq j < k \leq i/2 \\ j+k \geq 0 \\ k-j \leq n-i}} V_{\lambda_{i-2j} + \lambda_{i-2k}}$$

for all i and

$$V_{\lambda_i} \otimes V_{\lambda_j} \simeq \sum_{\substack{k, l \geq 0 \\ i-k-l \geq 0 \\ j+k \leq n}} V_{\lambda_{i-k-l} + \lambda_{j+k+l}}$$

for $i < j$ with $j-i$ even, as $U(\mathfrak{sp}_n)$ -modules.

⁶Only the type A is excluded in what follows

Proof. We invoke the rules for computing tensor products of representations of the symplectic group in terms of Young diagrams (these can be derived from [17, Sec. 2.5]). We will write a Young diagram as a nonincreasing sequence where the entries give the height of the subsequent *columns*. In particular $\mathbb{C} = [\]$, $V_{\lambda_i} = [i]$ and $V_{\lambda_i + \lambda_j} = [j, i]$ if $i \leq j$. For convenience, we allow ourselves to write $[i, 0]$ for $[i]$ and $[0]$ for $[\]$ (this mirrors the convention $\lambda_0 = 0$). Furthermore, a column of length $n + i$ is replaced by one of length $n - i$. If the Young diagrams were used to denote representations of \mathfrak{sl}_{2n} rather than \mathfrak{sp}_n , the rule for decomposing a tensor product $[j] \otimes [i]$ with $j \geq i$ would be simply:

$$[j] \otimes [i] = [j + i, 0] + [j + i - 1, 1] + \cdots + [j, i], \quad (\text{for } \mathfrak{sl}_{2n})$$

i.e., we put the columns $[j]$ (first, ‘red’) and $[i]$ (second, ‘black’) next to each other, and move a number black boxes underneath the red ones. Since in the case of \mathfrak{sp}_n we additionally have the invariant symplectic form on $[1]$, the rule should be modified so that when moving a red box, we can either ‘add’ it, appending to the first column, or ‘subtract’ it, annihilating a red box. We may assume we first add a number of black boxes, and then subtract a number of them. Furthermore, self-duality of $[1]$ implies that we ought to remove a red-black pair from the first column as soon as it becomes taller than n : in other words, we may add a black box only as long as the height of the first column is at most n . Thus we obtain

$$(26) \quad [j] \otimes [i] = \sum_{\substack{k, l \geq 0 \\ i - k - l \geq 0 \\ j + k \leq n}} [j + k - l, i - k - l] = \sum_{\substack{j - n \leq p \leq q \leq i, \\ p + q \geq 0 \\ q - p \leq 2(n - j)}} [j - p, i - q]$$

where the first sum clearly coincides with the original expression for $V_{\lambda_i} \otimes V_{\lambda_j}$. The same formula specialises to the symmetric square of $[i]$, where the terms of the above sum contained in $S^2[i] \subset [i] \otimes [i]$ are those of the form $[i - p, i - q]$ with p, q even. These are easily seen to give the original expression for $S^2 V_{\lambda_i}$ (with $j = p$ and $k = q$). \square

Remark 5. It is convenient to view equations 26 in the Grothendieck group K of the category of finite-dimensional $U(\mathfrak{sp}_n)$ -modules (this is simply the free abelian group generated by classes of finite-dimensional irreducible representations of \mathfrak{sp}_n). The relations may be then inverted so that, in particular, the class $[V_{2\lambda_n}]$ may be expressed as a linear combination of $[S^2 V_{\lambda_i}]$ and $[V_{\lambda_i} \otimes V_{\lambda_j}]$ with $0 \leq i < j \leq n$. However, since we are only interested in the dimension of the space of $\mathfrak{g}_0^{\text{ss}}$ -invariants, it is easier to first apply the corresponding homomorphism $K \rightarrow \mathbb{Z}$ to both sides of the above equations (viewed in K), and then solve for $\dim(V_{2\lambda_n})^{\mathfrak{g}_0^{\text{ss}}}$ in terms of modules whose invariants we know.

Lemma 20. *Let \bar{w}_o be the longest element in the Weyl group of $\mathfrak{g}_0^{\text{ss}}$, and $\bar{h}^\circ \in \mathfrak{h} \cap \mathfrak{g}_0^{\text{ss}}$ the sum of the coroots associated to positive roots of $\mathfrak{g}_0^{\text{ss}}$. Let $\iota : \mathfrak{h} \cap \mathfrak{g}_0^{\text{ss}} \rightarrow \mathfrak{h}$ denote the inclusion of Cartan subalgebras, and ι^* its transpose acting as restriction on weights. Then*

$$\begin{aligned} 2 \dim(S^2 V_{\lambda_i})^{\mathfrak{g}_0^{\text{ss}}} &= \#\{(w, w') \in W_i^{\mathfrak{p}} \times W_i^{\mathfrak{p}} \mid -\bar{w}_o \iota^*(w \cdot 0) = \iota^*(w' \cdot 0)\} \\ &+ \#\{w \in W_i^{\mathfrak{p}} \mid -\bar{w}_o \iota^*(w \cdot 0) = \iota^*(w \cdot 0), \quad \langle \iota^*(w \cdot 0), \bar{h}^\circ \rangle = 0 \pmod{2}\} \\ &- \#\{w \in W_i^{\mathfrak{p}} \mid -\bar{w}_o \iota^*(w \cdot 0) = \iota^*(w \cdot 0), \quad \langle \iota^*(w \cdot 0), \bar{h}^\circ \rangle = 1 \pmod{2}\}, \\ \dim(V_{\lambda_i} \otimes V_{\lambda_j})^{\mathfrak{g}_0^{\text{ss}}} &= \#\{(w, w') \in W_i^{\mathfrak{p}} \times W_j^{\mathfrak{p}} \mid -\bar{w}_o \iota^*(w \cdot 0) = \iota^*(w' \cdot 0)\}, \end{aligned}$$

for all $1 \leq i, j \leq n$.

Proof. Recall that $-\bar{w}_o$ sends the highest weight of a finite-dimensional simple $U(\mathfrak{g}_0^{\text{ss}})$ -module to the highest weight of its dual. In particular a finite-dimensional simple $U(\mathfrak{g}_0^{\text{ss}})$ -module of highest weight $\bar{\lambda}$ is self-dual if and only if $-\bar{w}_o \bar{\lambda} = \bar{\lambda}$. The self-duality is implemented either by a symmetric bilinear invariant or an alternating one. By [11, Thm. 3.2.17], the parity of the invariant coincides with the parity of $\langle \bar{\lambda}, \bar{h}^\circ \rangle$. The formulae then follow from Lemma 18. \square

We interpret the equations of Lemma 19 as relations

$$(27) \quad [S^2 V_{\lambda_i}] - \sum_{0 \leq j \leq i/2} [V_{2\lambda_{i-2j}}] - \sum_{\substack{\frac{i-n}{2} \leq j < k \leq \frac{i}{2} \\ j+k \geq 0 \\ k-j \leq n-i}} [V_{\lambda_{i-2j} + \lambda_{i-2k}}] = 0 \quad (1 \leq i \leq n)$$

$$[V_{\lambda_i} \otimes V_{\lambda_j}] - \sum_{\substack{k, l \geq 0 \\ i-k-l \geq 0 \\ j+k \leq n}} [V_{\lambda_{i-k-l} + \lambda_{j+k-l}}] = 0 \quad (1 \leq i < j \leq n, \quad j-i \in 2\mathbb{Z})$$

in K , with $\lambda_0 = 0$ by definition (as explained in the above Remark 5). Applying the homomorphism $K \rightarrow \mathbb{Z}$, $[M] \mapsto \dim M^{\mathfrak{g}_0^{\text{ss}}}$, and substituting the expressions given in Lemma 20, we obtain a determined linear system for the unknowns

$$d_i = \dim(V_{2\lambda_i})^{\mathfrak{g}_0^{\text{ss}}}, \quad 1 \leq i \leq n, \quad d_{ij} = \dim(V_{\lambda_i + \lambda_j})^{\mathfrak{g}_0^{\text{ss}}}, \quad 1 \leq i < j \leq n, \quad j-i \in 2\mathbb{Z}.$$

In particular, we may solve for d_n .

The only non-trivial task is the Weyl-group computation in Lemma 20. Let us recall that $W^{\mathfrak{p}}$ is the set of minimal length coset representatives for the quotient $W/W_{\mathfrak{p}}$, where $W_{\mathfrak{p}} \subset W$ denotes the parabolic Weyl subgroup generated by simple roots in Φ_0 . Equivalently, $W_{\mathfrak{p}}$ is the stabiliser of the longest root $\gamma \in \Phi^+$, so that $W^{\mathfrak{p}}$ may be naturally identified with the orbit $W\gamma$ (note that γ , being the heighest weight of the adjoint representation, is a fundamental weight since type A has been excluded). This gives rise to an algorithm for generating $W^{\mathfrak{p}}$ described, e.g., in [8, Prop. 3.2.16 and

the following paragraph]. Putting these together, we have the following algorithm to compute $\dim(V_{2\lambda_n})^{\mathfrak{g}_0^{\text{ss}}} = \dim R_2$ for \mathfrak{g} of type D, E or G and rank ℓ .

(1) Obtain from a database:

the Cartan matrix of Φ as a list of ℓ elements of \mathbb{Z}^ℓ ,
 the highest weight of \mathfrak{g} as an integer $1 \leq a \leq \ell$,
 the involution $-\bar{w}_o$ as a permutation of $\{1, \dots, \ell\}$ fixing a ,
 the coroot \bar{h}° as an element of \mathbb{Z}^ℓ with trivial a -th entry,
 the integer n ,

where we use \mathbb{Z}^ℓ to represent weights (in the basis of fundamental weights) and coroots (in the basis of simple coroots).

(2) Generate W_i^p , $1 \leq i \leq n$, as lists of words over $\{1, \dots, \ell\}$.

(3) Compute $\dim(S^2 V_{\lambda_i})^{\mathfrak{g}_0^{\text{ss}}}$, $1 \leq i \leq n$ and $\dim(V_{\lambda_i} \otimes V_{\lambda_j})^{\mathfrak{g}_0^{\text{ss}}}$, $1 \leq i < j \leq n$ as in Lemma 20.

(4) Set up the formal linear system (27) and substitute

$$[V_{\lambda_0}] \mapsto 1, [V_{\lambda_i}] \mapsto 1, [S^2 V_{\lambda_i}] \mapsto \dim(S^2 V_{\lambda_i})^{\mathfrak{g}_0^{\text{ss}}} \ (1 \leq i \leq n), \quad [V_{\lambda_i} \otimes V_{\lambda_j}] \mapsto \dim(V_{\lambda_i} \otimes V_{\lambda_j})^{\mathfrak{g}_0^{\text{ss}}} \ (1 \leq i < j \leq n),$$

and

$$[V_{2\lambda_i}] \mapsto d_i \ (1 \leq i \leq n), \quad [V_{\lambda_i + \lambda_j}] \mapsto d_{ij} \ (1 \leq i < j \leq n).$$

(5) Solve the resulting linear system on d_i, d_{ij} over \mathbb{Z} .

(6) Return d_n .

The algorithm is straightforward to implement (see [9] for a comprehensive discussion of computational methods in Lie theory). Note that since $\mathfrak{g}_0^{\text{ss}}$ is simply-laced, the coefficients of \bar{h}° in the basis of simple coroots coincide with those of the sum of all positive roots of $\mathfrak{g}_0^{\text{ss}}$ in the basis of simple roots.

4.8. Types E₆, E₇, E₈. We list the database entries required for the computation, and the final answer. The code used for this computation is available as [12]. The expressions for $-\bar{w}_o$ and \bar{h}° can be found in Bourbaki [4, §4, tables, entries VII and XI], up to the necessary relabelling the of Dynkin sub-diagram corresponding to $\mathfrak{g}_0^{\text{ss}} \subset \mathfrak{g}$ (the Bourbaki labelling of the diagram for \mathfrak{g} induces a labelling on the diagram of $\mathfrak{g}_0^{\text{ss}}$ that has to be mapped to its own Bourbaki labelling). We conclude that $\dim R_2 = 1$ in all three cases. Since the grading induced by the central torus of G_0 is a rescaling of the standard one (see Remark 2), we have that there exists a unique degree 2 element in $\text{Inv}(X, G)$.

E ₆	E ₇	E ₈
$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$
$a = 2$ $-\bar{w}_o = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 5 & 4 & 3 & 1 \end{pmatrix}$ $\bar{h}^\circ = (5, 0, 8, 9, 8, 5)$ $n = 10$	$a = 1$ $-\bar{w}_o = \text{id}$ $\bar{h}^\circ = (0, 15, 15, 28, 24, 18, 10)$ $n = 16$	$a = 8$ $-\bar{w}_o = \text{id}$ $\bar{h}^\circ = (34, 49, 66, 96, 75, 52, 27, 0)$ $n = 28$
$d_n = 1$	$d_n = 1$	$d_n = 1$

4.9. Type F₄. In order to deal with the remaining type F₄, we invoke the brute-force branching method relying on the computation of formal characters. We shall take for granted that a procedure for computing the formal character of a given finite-dimensional highest-weight module of a given semi-simple Lie algebra is at our disposal (these are typically refinements of the Freudenthal multiplicity formula, see [9, Sec. 8.9]). The algorithm to compute $\dim(V_{d\lambda_n})^{\mathfrak{g}_0^{\text{ss}}}$ is then as follows.

- (1) Identify the weight lattice of \mathfrak{sp}_n , resp. $\mathfrak{g}_0^{\text{ss}}$, with \mathbb{Z}^n , resp. \mathbb{Z}^3 , using the bases of fundamental weights.
- (2) Choose an element $w_i \in W_i^p$ for each $1 \leq i \leq n$.
- (3) Construct the \mathbb{Z} -module homomorphism $\rho : \mathbb{Z}^n \rightarrow \mathbb{Z}^3$ representing the map sending λ_i to $\iota^*(w_i \cdot 0)$.
- (4) Compute the formal character $\text{ch } V_{d\lambda_n}$ as an element of the group ring of \mathbb{Z}^n .
- (5) Compute $\rho_* \text{ch } V_{d\lambda_n}$, an element in the group ring of \mathbb{Z}^3 .
- (6) Decompose

$$(28) \quad \rho_* \text{ch } V_{d\lambda_n} = \sum c_\mu \text{ch } L_\mu$$

into a combination formal characters of finite-dimensional simple $U(\mathfrak{g}_0^{\text{ss}})$ -modules.

- (7) Return c_0 .

In our case we have $\mathfrak{g}_0^{\text{ss}}$ of type C_3 and, using Bourbaki's labelling of the fundamental weights, the matrix of ρ reads

$$\rho = \begin{pmatrix} 0 & 0 & 1 & 3 & 5 & 4 & 4 \\ 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The formal characters are too complex to be included here. A simple program computing the decomposition (28) in LiE is available as [13]. We conclude that $\dim(V_{d\lambda_n})^{\mathfrak{g}_0^{\text{ss}}}$ is zero for $1 \leq d \leq 3$ and one for $d = 4$, thus completing the proof of Theorem 2 in type F.

5. FURTHER DISCUSSION

5.1. Maximality. One interesting feature of G -invariant PDEs over X is that even their local infinitesimal symmetry algebras are precisely isomorphic to \mathfrak{g} . This had already been observed in [23]. We shall sketch the argument.

Lemma 21. *Assume \mathfrak{g} is not of type A. Then the map $\mathfrak{g}_0 \rightarrow \mathfrak{csp}(\mathfrak{g}_{-1})$ induced by the adjoint action is an embedding onto a maximal subalgebra.*

Inspecting the list of embeddings $\mathfrak{g}_0^{\text{ss}} \subset \mathfrak{sp}_n$, this may be extracted from Dynkin's classification of maximal subgroups in the simple Lie groups. Since we have already used Kostant's theorem in Subsection 4.6 above, we will use it here to provide a self-contained proof.

Proof. Following the notation introduced in the proof of the main Theorem 1, we work in $C_0^\bullet(\mathfrak{g}_-, \mathfrak{g})$ and suppress the $(\mathfrak{g}_-, \mathfrak{g})$ part from the notation. We have $C_0^0 = \mathfrak{g}_0$, $C_0^1 = \text{End}_0 \mathfrak{g}_-$, $Z_0^1 \simeq \mathfrak{csp}(\mathfrak{g}_{-1})$. Since $H^0(\mathfrak{g}_-, \mathfrak{g})$ is the annihilator of \mathfrak{g}_- in \mathfrak{g} , namely \mathfrak{g}_{-2} , we have $H_0^0 = 0$ whence $\mathfrak{g}_0 \rightarrow \mathfrak{csp}(\mathfrak{g}_{-1})$ is injective. Now, by Kostant's theorem, $H^1 \simeq L(s_\alpha \cdot \gamma)$, where $\gamma \in \Phi^+$ is the longest root, α is the unique simple root non-orthogonal to γ , and $L(\lambda)$ denotes the simple $U(\mathfrak{g}_0)$ -module with highest weight λ . In particular, we have just demonstrated that $H_0^1 \simeq \mathfrak{csp}(\mathfrak{g}_{-1})/\mathfrak{g}_0$ is a simple $U(\mathfrak{g}_0)$ -module (nil in type C), thus proving the claim. \square

Since we wish to speak of local symmetries of a PDE, let us first point out that the construction of the bundle of Lagrangian Grassmannians (as in (1)) is functorial with respect to local contactomorphisms. More precisely, if $\phi : U \rightarrow V$ is a map of contact complex manifolds inducing isomorphisms on fibres of contact distributions, then ϕ lifts naturally to a map $\phi^{(1)} : U^{(1)} \rightarrow V^{(2)}$ inducing isomorphisms on fibres. It will be convenient to work with germs of infinitesimal symmetries at a point. Let us quickly define the necessary terms.

Definition 8. *Let $\mathcal{E} \subset X^{(1)}$ be a hypersurface, $x \in X$ a point and $[v]$ a germ at x of a vector field on X . We say that $[v]$ is a germ of an infinitesimal symmetry of \mathcal{E} if there exist: open neighbourhoods $V \subset U$ of x , an open disc $\Delta \subset \mathbb{C}$ and a representative v of $[v]$ on U , such that*

- (1) v is contact, i.e. $[v, \mathcal{C}] \subset \mathcal{C}$ over U ,
- (2) the local flow $\phi_t : V \rightarrow U$, $t \in \Delta$, of v is well-defined,
- (3) the lifts $\phi_t^{(1)} : V^{(1)} \rightarrow U^{(1)}$, $t \in \Delta$, preserve \mathcal{E} .

Proposition 7. *Assume \mathfrak{g} is not of type A.⁷ Let $\mathcal{E} \in \text{Inv}(X, G)$ be a G -invariant hypersurface in $X^{(1)}$. Fix a point $x \in X$ and let \mathfrak{s} be the Lie algebra of germs at x of infinitesimal symmetries of \mathcal{E} . Then the local action map $\mathfrak{g} \rightarrow \mathfrak{s}$ is an isomorphism.*

The proof is a standard application of Tanaka theory [22] (see also [19]).

Proof. By G -invariance we may take $x = o$ and use the identification $T_o X \simeq \mathfrak{g}/\mathfrak{g}^0$. Let $\text{ev} : \mathfrak{s} \rightarrow \mathfrak{g}/\mathfrak{g}^0$ be the evaluation map at o . There is a natural filtration on \mathfrak{s} defined by setting $\mathfrak{s}^i = \text{ev}^{-1} \mathfrak{g}^i/\mathfrak{g}^0$ for $i \leq 0$, and extending inductively for $i > 0$ so that $v \in \mathfrak{s}^{i+1}$ if and only if $[v, \mathfrak{s}^{-1}] \subset \mathfrak{s}^i$. It is straightforward to check that $[\mathfrak{s}^i, \mathfrak{s}^j] \subset \mathfrak{s}^{i+j}$ and thus we have the associated graded Lie algebra $\text{gr } \mathfrak{s}$. Furthermore, the natural embedding $\mathfrak{g} \rightarrow \mathfrak{s}$ is a homomorphism of filtered Lie algebras. Since $\mathfrak{g} \subset \mathfrak{s}$ acts infinitesimally transitively, we have that $\text{ev} : \mathfrak{s}/\mathfrak{s}^0 \rightarrow \mathfrak{g}/\mathfrak{g}^0$ is an isomorphism of vector spaces. We now observe that:

- (1) the induced map $\text{gr}_- \mathfrak{s} \rightarrow \text{gr}_- \mathfrak{g}$ is an isomorphism of graded nilpotent Lie algebras,
- (2) the maps $\text{gr}_i \mathfrak{s} \rightarrow \text{Hom}(\text{gr}_{-1} \mathfrak{s}, \text{gr}_{i-1} \mathfrak{s})$ induced by the adjoint action are injective.

Claim (1) is most easily seen by restricting to $\text{gr}_- \mathfrak{g} \subset \text{gr}_- \mathfrak{s}$. Claim (2) follows from the very definition of the filtration. We will show that $\mathfrak{g}_i \rightarrow \text{gr}_i \mathfrak{s}$ is an isomorphism. This is clearly true for $i < 0$, and also for $i = 0$ by Lemma 21. By induction we may assume it had been shown for all $i < k$, $k > 0$. We then have $\text{gr}_k \mathfrak{s} \subset \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{k-1})$ by (2) above and by the inductive hypothesis. In fact, using the Lie algebra structure we may extend this to an embedding of $\text{gr}_k \mathfrak{s}$ into the space $\text{Der}_k(\mathfrak{g}_-, \mathfrak{g})$ of degree k derivations of \mathfrak{g}_- into the $U(\mathfrak{g}_-)$ -module \mathfrak{g} . This space clearly contains \mathfrak{g}_k and we have

$$\text{Der}_k(\mathfrak{g}_-, \mathfrak{g})/\mathfrak{g}_k \simeq H_k^1(\mathfrak{g}_-, \mathfrak{g}).$$

It will thus be enough to check that the cohomology space on the right hand side vanishes for $k > 0$. This had been done by Yamaguchi in [26, Prop. 5.1 (2)]. \square

⁷Nor of type C, see Remark 1.

5.2. The Lagrangian Chow transform and invariant hypersurfaces of geometric origin. As we have already remarked, one may produce certain invariant PDEs over adjoint varieties by means of a straightforward geometric construction. This observation is a key idea in [23]. We review it here for purpose of comparison. As before, \mathfrak{g} is simple not of type C, with adjoint variety $X \subset \mathbb{P}\mathfrak{g}$. The notation is as introduced in Sections 2 and 3. In particular, the contact hyperplane at the origin $o \in X$ is identified with \mathfrak{g}_{-1} , and the action of the isotropy subgroup $P \subset G$ of o restricted to \mathfrak{g}_{-1} factors through the reductive group G_0 (see Subsection 3.1).

Definition 9. *The sub-adjoint variety $Y \subset \mathbb{P}\mathfrak{g}_{-1}$ of \mathfrak{g} is the union of the closed orbits of G_0 in the projectivised irreducible summands of \mathfrak{g}_{-1} .*

Of course, as indicated in the table in Lemma 3, the only case with decomposable \mathfrak{g}_{-1} is type A: there $G_0 \simeq \mathrm{GL}_n \times \mathbb{C}^\times$, $\mathfrak{g}_{-1} \simeq \mathbb{C}^n \oplus \mathbb{C}^{n*}$ and $Y \simeq \mathbb{P}^{n-1} \sqcup \mathbb{P}^{(n-1)*}$. In the remaining cases \mathfrak{g}_{-1} is irreducible, and so is Y . As we will soon explain, it is interesting to compute the degree of Y as a subvariety of $\mathbb{P}\mathfrak{g}_{-1}$.

Lemma 22. *The following table, supplementing that of Lemma 3, gives the sub-adjoint variety $Y \subset \mathbb{P}\mathfrak{g}_{-1}$ and its degree. The sub-adjoint variety is always Legendrian and, in particular, of codimension n .*

restriction	type of \mathfrak{g}	\mathfrak{g}_0^{ss}	\mathfrak{g}_{-1}	Y	embedding	deg Y
$n \geq 1$	A_{n+1}	\mathfrak{sl}_n	$\mathbb{C}^n \oplus \mathbb{C}^{n*}$	$\mathbb{P}^{n-1} \sqcup \mathbb{P}^{(n-1)*}$	linear	2
$n \geq 3$	$B_{(n+3)/2}$ or $D_{(n+4)/2}$	$\mathfrak{sl}_2 \oplus \mathfrak{so}_n$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathbb{P}^1 \times Q^{n-2}$	Segre	$2(n-1)$
	E_6	\mathfrak{sl}_6	$\Lambda^3 \mathbb{C}^6$	$\mathrm{Gr}(3, 6)$	Plücker	42
	E_7	\mathfrak{spin}_{12}	spinor 32-dim	15-dim spinor		286
	E_8	E_7	fundamental 56-dim	27-dim Freudenthal		13188
	F_4	\mathfrak{sp}_3	$\Lambda_0^3 \mathbb{C}^6$	$\mathrm{LGr}(3, 6)$	Plücker	16
	G_2	\mathfrak{sl}_2	$S^3 \mathbb{C}^2$	\mathbb{P}^1	Veronese	3

Here $Q^{n-2} \subset \mathbb{P}^{n-1}$ denotes a non-singular quadric hypersurface.

Proof. The description of Y , its embedding in $\mathbb{P}\mathfrak{g}_{-1}$ and the Legendrian property may be found in [7]. The degrees in types A, B, D and G_2 are easy to compute. For A we have the union of a pair of linear varieties of equal dimension, hence of degree $1 + 1 = 2$. For G_2 we have the twisted cubic $\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$, hence of degree 3 ($\nu_d^* \mathcal{O}(1) = \mathcal{O}(d)$ for the d^{th} Veronese embedding). Finally for B and D we have the Segre-embedded product of a line times a quadric. Computing in the Chow ring, we have $A(Y) = A(\mathbb{P}^1) \otimes A(Q^{n-2})$ and in particular $A^{n-1}(Y) = A^1(\mathbb{P}^1) \otimes A^{n-2}(Q^{n-2})$ generated by $[\mathrm{pt}] \otimes [\mathrm{pt}]$; we denote by $h \in A^1(Q^{n-2})$ the class of the hyperplane section so that $h^{n-2} = 2[\mathrm{pt}]$. Now, the pullback of the hyperplane class by the Segre embedding is $[\mathrm{pt}] \otimes 1 + 1 \otimes h$, and we compute the degree as

$$([\mathrm{pt}] \otimes 1 + 1 \otimes h)^{n-1} = (n-1)([\mathrm{pt}] \otimes 1)(1 \otimes h)^{n-2} = 2(n-1)[\mathrm{pt}] \otimes [\mathrm{pt}].$$

The degree of the Lagrangian Grassmannian in its Plücker embedding is given as the very first formula in [25], leading in our case to $2^3 6! 2! / 3! 5! = 16$. It remains to find the degrees in types E and F_4 . The necessary information can be extracted from the paper [15], to which we keep referring in what follows. More precisely, we are interested in the answer to Problem 2.3 on p. 47 for the following pairs (G, P_α) :

$$(A_5, P_3), (D_6, P_6), (E_7, P_7)$$

with Bourbaki labelling as usual. The value $42 = 9! / 2^2 3^3 4^2 5$ for the Grassmannian SL_6 / P_3 is given by the Example on p. 46 ($n = k = 3$). As explained on p. 51, the pair (D_6, P_6) may be replaced by (B_5, P_5) and then the value $286 = 15! 2! 4! / 5! 7! 8! 9!$ is given by Corollary 4.9 on p. 54 ($n = 5, d = 15$). The value 13188 for the Freudenthal variety E_7 / P_7 appears in the ‘Remarks’ on p. 57. \square

The idea is now to produce a hypersurface $\mathcal{E}_Y \subset \mathrm{LGr} \mathfrak{g}_{-1}$ from the sub-adjoint variety $Y \subset \mathbb{P}\mathfrak{g}_{-1}$. This is a Lagrangian version of the usual Chow form, assigning to a projective variety of codimension k in \mathbb{P}^{N-1} a hypersurface in the Grassmannian $\mathrm{Gr}(k, N)$. Let us state its properties.

Lemma 23. *Fix a standard symplectic form on \mathbb{C}^{2n} and consider the canonical Sp_n -equivariant double fibration*

$$(29) \quad \begin{array}{ccc} & \mathrm{Fl}^{\mathrm{iso}}(1, n, 2n) & \\ p \swarrow & & \searrow q \\ \mathbb{P}^{2n-1} & & \mathrm{LGr}(n, 2n), \end{array}$$

where $\mathrm{Fl}^{\mathrm{iso}}(1, n, 2n)$ is the isotropic partial flag variety embedded into $\mathbb{P}^{2n-1} \times \mathrm{LGr}(n, 2n)$ as an incidence correspondence. Let $Z \subset \mathbb{P}^{2n-1}$ be an irreducible subvariety of pure codimension n and degree d . Then $\mathcal{E}_Z = q(p^{-1}Z)$ is an irreducible hypersurface of degree d in $\mathrm{LGr}(n, 2n)$.

Definition 10. *We call \mathcal{E}_Z the Lagrangian Chow transform of Z . More generally, if Z_1, \dots, Z_r are several irreducible components, all of codimension n , we set $\mathcal{E}_{\bigcup Z_i} = q(p^{-1} \bigcup Z_i) = \bigcup \mathcal{E}_{Z_i}$.*

Proof. The projection p is a Zariski-locally trivial bundle with fibres isomorphic to $\mathrm{LGr}(n-1, 2(n-1))$. It then follows immediately that \mathcal{E}_Z is irreducible. Furthermore, $\dim \mathcal{E}_Z \leq \dim p^{-1}Z = \dim Z + (n-1)n/2 = (n^2 + n - 2)/2$ so that $\mathrm{codim} \mathcal{E}_Z \geq 1$. Now, consider a general line $\ell \subset \mathrm{LGr}(n, 2n)$: there is a canonical identification $\ell \simeq \mathbb{P}(K/K^\perp)$ where

$K \subset \mathbb{C}^{2n}$ is a general $(n+1)$ -dimensional subspace such that the symplectic form restricted to K has rank 1. Since $\mathbb{P}K$ has complementary dimension to Z , the intersection $\mathbb{P}K \cap Z$ is nonempty. So is then $\ell \cap \mathcal{E}_Z$, proving $\text{codim } \mathcal{E}_Z = 1$. Now, ℓ being general, its intersection with \mathcal{E}_Z is transverse and consists of $\deg \mathcal{E}_Z$ points. That is, $\mathbb{P}K^\perp \cap Z = \emptyset$ and we may identify $\mathbb{P}K \cap Z$ with $\ell \cap \mathcal{E}_Z$ scheme-theoretically: in particular, $\mathbb{P}K \cap Z$ is reduced and thus consists of d points, proving $\deg \mathcal{E}_Z = d$. \square

This way the sub-adjoint variety gives rise to a hypersurface in the Lagrangian Grassmannian. Furthermore, since the construction is manifestly Sp_n -invariant, it follows that $\mathcal{E}_Y \subset \text{LGr } \mathfrak{g}_{-1}$ is G_0 -invariant. Its degree is equal to that of Y (in the only case where Y is not irreducible, the transform of each component is a distinct hyperplane section of $\text{LGr } \mathfrak{g}_{-1}$, so that their union has degree 2), thus explaining the last row in the table we have included in Theorem 1. In particular, it is remarkable that, except for type A and G_2 , the ‘natural’, geometric \mathcal{E}_Y is of very high degree, as compared to the minimal degree hypersurfaces we have produced. On the other hand, in types A and G_2 , the components of the hypersurface \mathcal{E}_Y are the unique invariant hypersurfaces and thus coincide with ours.

6. EXPLICIT INVARIANT PDES

The key to recast our main result, that is Theorem 1, in the context of nonlinear PDEs is to choose suitable Darboux coordinates on the adjoint variety $X = G/P$. Recall also Lemma 5.

Proposition 8. *For any bi-Lagrangian decomposition $L \oplus L^* = \mathfrak{g}_{-1}$ there exist complex Darboux coordinates*

$$(30) \quad x^1, \dots, x^n, u, u_1, \dots, u_n,$$

in a neighborhood of o , such that

$$L = g^{-1} \cdot \left\langle D_{x^1}|_{gP}, \dots, D_{x^n}|_{gP} \right\rangle, \quad \mathfrak{g}_{-2} = g^{-1} \cdot \left\langle \frac{\partial}{\partial u}|_{gP} \right\rangle, \quad L^* = g^{-1} \cdot \left\langle \frac{\partial}{\partial u_1}|_{gP}, \dots, \frac{\partial}{\partial u_n}|_{gP} \right\rangle,$$

for all $g \in G$, where D_{x^i} are the total derivatives (cf. (2)).

Proof. By restricting the exponential map $\mathfrak{g} \ni g \mapsto \exp(g) \in G$ to \mathfrak{g}_- , one obtains an (algebraic) isomorphism $\Psi : \mathfrak{g}_- \rightarrow U \subseteq X$, between the linear space \mathfrak{g}_- and an open neighborhood U of the origin. For $v \in \mathfrak{g}_-$ denote by \hat{v} the vector field on U induced by v . Then we have that

$$(31) \quad \hat{v}_{\Psi(w)} = T_w \Psi \left(v - \frac{1}{2}[w, v] \right),$$

for all $w \in \mathfrak{g}_-$. Formula (31) follows directly from the Baker–Campbell–Hausdorff formula

$$e^{t\hat{v}} \Psi(w) = e^{tv} e^w P = e^{tv+w+\frac{1}{2}[tv,w]} P = \Psi \left(w + tv + \frac{1}{2}t[v, w] \right).$$

We can now pull-back the contact distribution \mathcal{C} on X to a contact distribution (denoted by the same symbol \mathcal{C}) on \mathfrak{g}_- , by setting $\mathcal{C}_w := (T_w \Psi)^{-1}(\mathcal{C}_{\Psi(w)})$. Then (31) implies

$$(32) \quad \mathcal{C}_w = \left(\text{id} - \frac{1}{2} \text{ad}_w \right) \mathfrak{g}_{-1}.$$

Fix now vectors $l^1, \dots, l^n, r, l_1, \dots, l_n$ such that $L = \langle l^1, \dots, l^n \rangle$, $\mathfrak{g}_{-2} = \langle r \rangle$, $L^* = \langle l_1, \dots, l_n \rangle$. Then from (32) it follows that the vectors fields D_i and V^i on \mathfrak{g}_{-1} defined by

$$D_i|_w := \left(\text{id} - \frac{1}{2} \text{ad}_w \right) l_i, \quad V^i|_w := \left(\text{id} - \frac{1}{2} \text{ad}_w \right) l^i,$$

for all $w \in \mathfrak{g}_-$, form a basis of \mathcal{C} . The last step is to show that there are coordinates

$$(33) \quad \underline{x}^1, \dots, \underline{x}^n, \underline{u}, \underline{u}_1, \dots, \underline{u}_n,$$

on \mathfrak{g}_- such that

$$D_i = \frac{\partial}{\partial \underline{x}^i} + \underline{u}_i \frac{\partial}{\partial \underline{u}}, \quad V^i = \frac{\partial}{\partial \underline{u}^i}.$$

But this is true, if one sets $\underline{x}^i := l^i$, $\underline{u} := r^\vee + l^i l_i$, $\underline{u}_i := l_i$. Then the desired coordinates (30) are just the pull-backs via Ψ of (33). \square

6.1. The case A. Besides being technically the simplest, this case is made interesting by the fact that the torus T has rank 2 (see Remark 2). The adjoint manifold X is the projectivised cotangent bundle $\mathbb{P}T^*\mathbb{P}^{n+1}$, which is a $\mathrm{PGL}(n+2)$ -homogeneous contact manifold of dimension $2n+1$ (see Lemma 15). The contact plane \mathcal{C}_o at the origin is the direct sum $\mathbb{C}^n \oplus \mathbb{C}^{n*}$, which happens to be bi-Lagrangian. Hence, we choose Darboux coordinates as in Proposition 8, in such a way that \mathbb{C}^n is spanned by the ‘total derivatives’ D_{x^i} and \mathbb{C}^{n*} is spanned with the ‘vertical vectors’ ∂_{u_i} .

The fact that $\mathrm{rk} T = 2$ is mirrored by the fact that the sub-adjoint variety Y has two irreducible components: \mathbb{P}^{n-1} and $\mathbb{P}^{(n-1)*}$ (see Lemma 3). Accordingly, the Lagrangian Chow transform \mathcal{E}_Y of the *whole* Y is a hypersurface of degree 2, made by two irreducible components of degree 1, namely the Lagrangian Chow transforms $\mathcal{E}_{\mathbb{P}^{n-1}}$ and $\mathcal{E}_{\mathbb{P}^{(n-1)*}}$ of the corresponding irreducible pieces of Y (see Definition 10). Hence, $\mathcal{E}_{\mathbb{P}^{n-1}}$ and $\mathcal{E}_{\mathbb{P}^{(n-1)*}}$ are *both* homogeneous 2nd order PDEs in type A_{n+1} , of minimal degree.

In the Darboux coordinates provided by Proposition 8, these are precisely the parabolic Monge–Ampère equations, already discussed in [3]. For instance, in order to compute $\mathcal{E}_{\mathbb{P}^{n-1}}$, we just observe that a Lagrangian n -plane

$$(34) \quad L(u_{ij}) := \langle D_{x^i} + u_{ij}\partial_{u_j} \mid i = 1, \dots, n \rangle$$

intersects nontrivially $\mathbb{C}^n = \langle D_{x^i} \mid i = 1, \dots, n \rangle$ if and only if $\det(u_{ij}) = 0$. Hence, $\mathcal{E}_{\mathbb{P}^{n-1}} = \{\det(u_{ij}) = 0\}$. The other component $\mathcal{E}_{\mathbb{P}^{(n-1)*}}$ of \mathcal{E}_Y cannot be written explicitly as a 2nd order PDE, since no Lagrangian n -plane nontrivially intersecting \mathbb{C}^{n*} can be written in the form (34). Indeed, these Darboux coordinates are adapted to the structure given by the bilagrangian splitting of the contact distribution (there is a class of adapted coordinates, stable under PGL_{n+1} acting locally as point transformations). However, one could consider more ‘generic’ Darboux coordinates and simultaneously express both $\mathcal{E}_{\mathbb{P}^{n-1}}$ and $\mathcal{E}_{\mathbb{P}^{(n-1)*}}$ as explicit PDEs.

6.2. The case B₃. We denote by $Y \subset \mathbb{P}(\mathfrak{g}_{-1})$ the sub-adjoint variety of X . In this section we compute the Lagrangian Chow transform \mathcal{E}_Y of Y in the case $G = B_3$, because \mathcal{E}_Y is precisely the minimal-degree homogeneous equation on X . Observe that, as a second-order PDEs, \mathcal{E}_Y has 3 independent variables ($n = 3$), and as an algebraic hypersurface is of degree 4 in the minors of (u_{ij}) (recall Definition 7 (2)).

Before focusing on $G = B_3$, let us examine the general case when G is of type B or D. Let $\mathcal{C} = \langle D_{x^1}, \dots, D_{x^n}, \partial_{u_1}, \dots, \partial_{u_n} \rangle$ be the contact distribution on X , and identify $\mathcal{C}_o \equiv \mathfrak{g}_{-1} \equiv \mathbb{C}^2 \otimes \mathbb{C}^n = \langle A \otimes e_1, \dots, A \otimes e_n, B \otimes e_1, \dots, B \otimes e_n \rangle$, where $\mathbb{C}^2 = \langle A, B \rangle$ and $\mathbb{C}^n = \langle e_1, \dots, e_n \rangle$. Now recall that \mathbb{C}^n is equipped with a metric g , and that the sub-adjoint variety Y is the image of $\mathbb{P}^1 \times \mathcal{N}_g$ in the Segre embedding

$$\mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^n) \equiv \mathbb{P}^1 \times \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{2n-1} = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^n),$$

where $\mathcal{N}_g \subset \mathbb{P}(\mathbb{C}^n)$ is the null variety of g (see Subsection 4.5). We assume that g is diagonal, i.e., $g = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then Y has dimension $n-1$ in $\mathbb{P}(\mathfrak{g}_{-1}) = \mathbb{P}^{2n-1}$ (see Subsection 5.2), and the n quadratic equations

$$(35) \quad \mathrm{rank} \begin{pmatrix} dx^1 & dx^2 & \dots & dx^n \\ du_1 & du_2 & \dots & du_n \end{pmatrix} \leq 1, \quad \sum_{i=1}^n \lambda_i (dx^i)^2 = 0$$

vanish on Y . Indeed, the rank of the above $2 \times n$ matrix is ≤ 1 if and only if $n-1$ minors of rank two vanishes. Then, if an element $[v]$ of \mathbb{P}^{2n-1} is in the image of the Segre variety, it will be also in Y if and only if v is null with respect to the metric

$$\begin{pmatrix} 0 & \mathrm{diag}(\lambda_1, \dots, \lambda_n) \\ \mathrm{diag}(\lambda_1, \dots, \lambda_n) & 0 \end{pmatrix}$$

on \mathfrak{g}_{-1} , induced from the metric g . Let us now specialise to the case $n = 3$. If $\lambda_i = 1, \forall i$, the system (35) is

$$(36) \quad \begin{cases} x^1 u_2 - u_1 x^2 = 0 \\ x^1 u_3 - u_1 x^3 = 0 \\ x^1 u_1 + x^2 u_2 + x^3 u_3 = 0. \end{cases}$$

The variety $\tilde{Y} \subset \mathbb{P}^5$, cut out by (36), contains by construction the subadjoint variety Y , but also other ‘parasitic components’, namely

$$(37) \quad \begin{cases} u_1 = 0 \\ u_2 = 0 \\ u_3 = 0 \end{cases}, \quad \begin{cases} x^1 = 0 \\ u_1 = 0 \\ x^2 u_2 + x^3 u_3 = 0. \end{cases}$$

The first one is a \mathbb{P}^2 inside \mathbb{P}^5 , whereas the second is a quadric inside a \mathbb{P}^3 inside \mathbb{P}^5 .

By computing the equation $\mathcal{E}_{\tilde{Y}}$ associated with the variety \tilde{Y} described by (36), we obtain

$$\det(u_{ij}) \cdot (u_{13}^2 u_{22} + u_{12}^2 u_{33} - 2u_{12} u_{13} u_{23}) \cdot F = 0,$$

where F is a long expression of degree 6 in second derivatives. According to the general theory of Lagrangian Chow transforms, $\mathcal{E}_{\tilde{Y}}$ is composed of various irreducible components, only one of whose is the desired equation \mathcal{E}_Y (see Definition 10). It remains to establish which is which.

First, $\det(u_{ij}) = 0$ is the (Monge–Ampère) equation associated to the first variety of (37) in the spirit of [3], and it cannot be \mathcal{E}_Y since it is of degree 1 in the Plücker coordinates. Second, $u_{13}^2 u_{22} + u_{12}^2 u_{33} - 2u_{12} u_{13} u_{23} = 0$ is the equation associated to the second variety of (37). Now we obtain \mathcal{E}_Y in a less direct way, by computing the full ideal

$\mathcal{I}(Y)$ of Y . As it turns out, $\mathcal{I}(Y)$ is generated by 6 elements, as opposed to the 3 equations appearing in (36). To this end, recall that, by the Cauchy decomposition formula,

$$\begin{aligned} S^2(\mathbb{C}^2 \otimes \mathbb{C}^3) &= \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}(\mathbb{C}^2) \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}(\mathbb{C}^3) \\ &\quad \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array}(\mathbb{C}^2) \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array}(\mathbb{C}^3) \\ (38) \quad &= (S^2(\mathbb{C}^2) \otimes S^2(\mathbb{C}^3)) \oplus \mathbb{C}^3, \end{aligned}$$

where the boxes denote the appropriate Schur functors.

Dual to the Segre embedding

$$\begin{aligned} \mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^3) \equiv \mathbb{P}^1 \times \mathbb{P}^2 &\longrightarrow \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3) \equiv \mathbb{P}^5, \\ ([v], [w]) &\longmapsto [v \otimes w], \end{aligned}$$

there is the projection

$$(39) \quad S^2(\mathbb{C}^2 \otimes \mathbb{C}^3) \longrightarrow S^2(\mathbb{C}^2) \otimes S^2(\mathbb{C}^3)$$

from the 21-dimensional space $S^2(\mathbb{C}^2 \otimes \mathbb{C}^3)$ of quadratic forms on \mathbb{P}^5 , to the 18-dimensional space $S^2(\mathbb{C}^2) \otimes S^2(\mathbb{C}^3)$ of bi-homogeneous forms on the product space $\mathbb{P}^1 \times \mathbb{P}^2$, of bi-degree $(2, 2)$. The kernel of (39) is precisely the space \mathbb{C}^3 appearing in (38), which consists of the three quadrics cutting out the Segre in \mathbb{P}^5 .

Fix coordinates $[A : B]$ on \mathbb{P}^1 , and coordinates $[z^1 : z^2 : z^3]$ on \mathbb{P}^2 . Suppose that \mathbb{C}^3 is equipped with a non-degenerate quadratic form

$$(40) \quad g = (z^1)^2 + (z^2)^2 + (z^3)^2,$$

and let $Q := \{g = 0\} \subset \mathbb{P}^2$. Accordingly, we can single out the trace-free part of the quadratic forms on \mathbb{C}^3 , that is $S^2(\mathbb{C}^3) = S_0^2(\mathbb{C}^3) \oplus \langle g \rangle$, and thus (38) can be further split as

$$S^2(\mathbb{C}^2 \otimes \mathbb{C}^3) = ((S^2(\mathbb{C}^2) \otimes S_0^2(\mathbb{C}^3)) \oplus \mathbb{C}^3) \oplus ((S^2(\mathbb{C}^2) \otimes \langle g \rangle) \oplus \mathbb{C}^3).$$

The canonical projection $S^2(\mathbb{C}^2 \otimes \mathbb{C}^3) \longrightarrow (S^2(\mathbb{C}^2) \otimes S_0^2(\mathbb{C}^3)) \oplus \mathbb{C}^3$ is precisely the dual to the embedding $Y := \mathbb{P}^1 \times Q \longrightarrow \mathbb{P}^5$, and its 6-dimensional kernel, i.e.,

$$(41) \quad (S^2(\mathbb{C}^2) \otimes \langle g \rangle) \oplus \mathbb{C}^3,$$

is the space of quadrics cutting out the 2-fold Y in \mathbb{P}^5 . From (40) it follows immediately that

$$\begin{aligned} S^2(\mathbb{C}^2) \otimes \langle g \rangle &= \langle A^2, AB, B^2 \rangle \otimes \langle g \rangle = \langle A^2g, ABg, B^2g \rangle \\ &= \langle (Az^1)^2 + (Az^2)^2 + (Az^3)^2, Az^1Bz^1 + Az^2Bz^2 + Az^3Bz^3, (Bz^1)^2 + (Bz^2)^2 + (Bz^3)^2 \rangle \\ &= \langle (x^1)^2 + (x^2)^2 + (x^3)^2, x^1u_1 + x^2u_2 + x^3u_3, u_1^2 + u_2^2 + u_3^2 \rangle, \end{aligned}$$

having set

$$x^i := Az^i, \quad u_i := Bz^i.$$

In these coordinates, the \mathbb{C}^3 appearing in (41) is the 3-dimensional space $\langle x^1u_2 - x^2u_1, x^1u_3 - x^3u_1, x^2u_3 - x^3u_2 \rangle$, so that the ideal

$$(42) \quad \mathcal{I}(Y) = \langle x^1u_2 - x^2u_1, x^1u_3 - x^3u_1, x^2u_3 - x^3u_2, (x^1)^2 + (x^2)^2 + (x^3)^2, x^1u_1 + x^2u_2 + x^3u_3, u_1^2 + u_2^2 + u_3^2 \rangle$$

is generated by 6 elements, and this number cannot be reduced. Recall that $\deg Y = 4$ in virtue of Lemma 23. Now we find a generator of the ideal of $\mathcal{E}_Y := \{L \in \text{LGr}(\mathfrak{g}_{-1}) \mid L \cap Y \neq \emptyset\}$ (see Definition 10).

To this end we recall the standard coordinates u_{ij} on $\text{LGr}(\mathfrak{g}_{-1})$, introduced earlier in Subsection 1.3 (see also Lemma 5 (3)). Indeed, a Lagrangian plane L is locally given by the 3 equations

$$(43) \quad u_i = u_{ij}x^j, \quad i = 1, 2, 3,$$

where we recall that $u_{ij} = u_{ji}$. Moreover, L belongs to \mathcal{E}_Y if and only if, by replacing u_1, u_2 and u_3 according to (43), in each of the six generators of the ideal $\mathcal{I}(Y)$ given in (42), one obtains a compatible systems of polynomials in the three variables x^1, x^2, x^3 . These are:

$$\begin{aligned} q_1 &= x^1(u_{12}x^1 + u_{22}x^2 + u_{23}x^3) - x^2(u_{11}x^1 + u_{12}x^2 + u_{13}x^3), \\ q_2 &= x^1(u_{13}x^1 + u_{23}x^2 + u_{33}x^3) - x^3(u_{11}x^1 + u_{12}x^2 + u_{13}x^3), \\ q_3 &= x^2(u_{13}x^1 + u_{23}x^2 + u_{33}x^3) - x^3(u_{12}x^1 + u_{22}x^2 + u_{23}x^3), \\ q_4 &= u_{11}(x^1)^2 + 2u_{12}x^2x^1 + u_{22}(x^2)^2 + u_{33}(x^3)^2 + 2x^3(u_{13}x^1 + u_{23}x^2), \\ q_5 &= (x^1)^2 + (x^2)^2 + (x^3)^2, \\ q_6 &= (u_{11}x^1 + u_{12}x^2 + u_{13}x^3)^2 + (u_{12}x^1 + u_{22}x^2 + u_{23}x^3)^2 + (u_{13}x^1 + u_{23}x^2 + u_{33}x^3)^2. \end{aligned}$$

Here we identify polynomials in the u_{ij} with element of the affine coordinate ring of the corresponding open subset of $\text{LGr } \mathfrak{g}_{-1}$ (see the proof of Lemma 17). We furthermore invert x^3 to remove the apex of the affine cone over Y , so

that we may now use elimination to intersect the ideal generated by q_1, \dots, q_6 with the ring $\mathbb{C}[x^3, 1/x^3, u_{ij}]$. Using Macaulay 2, we obtain that the ideal of \mathcal{E}_Y in $\mathbb{C}[u_{ij}]$ pulls back to

$$(44) \quad \langle (x^3)^2 \cdot F \rangle \subset \mathbb{C}[x^3, 1/x^3, u_{ij}],$$

where

$$\begin{aligned} F = & 4u_{12}^6 + u_{11}^2 u_{12}^4 + 12u_{13}^2 u_{12}^4 + u_{22}^2 u_{12}^4 + 12u_{23}^2 u_{12}^4 - 8u_{33}^2 u_{12}^4 - 10u_{11} u_{22} u_{12}^4 + 8u_{11} u_{33} u_{12}^4 + 8u_{22} u_{33} u_{12}^4 - 36u_{11} u_{13} u_{23} u_{12}^3 - 36u_{13} u_{22} u_{23} u_{12}^3 \\ & + 72u_{13} u_{23} u_{33} u_{12}^3 + 12u_{13}^4 u_{12}^2 + 12u_{23}^4 u_{12}^2 + 4u_{33}^4 u_{12}^2 - 2u_{11} u_{22}^2 u_{12}^2 - 8u_{11} u_{33}^2 u_{12}^2 - 8u_{22} u_{33}^2 u_{12}^2 + 2u_{11}^2 u_{13}^2 u_{12}^2 + 8u_{11}^2 u_{22}^2 u_{12}^2 + 20u_{13}^2 u_{22}^2 u_{12}^2 \\ & + 20u_{11}^2 u_{23}^2 u_{12}^2 - 84u_{13}^2 u_{23}^2 u_{12}^2 + 2u_{22}^2 u_{23}^2 u_{12}^2 - 2u_{11} u_{22} u_{23}^2 u_{12}^2 + 2u_{11}^2 u_{33}^2 u_{12}^2 + 20u_{13}^2 u_{33}^2 u_{12}^2 + 2u_{22}^2 u_{33}^2 u_{12}^2 + 20u_{11}^2 u_{23}^2 u_{12}^2 + 20u_{11} u_{22} u_{33}^2 u_{12}^2 \\ & - 2u_{11}^3 u_{22}^2 u_{12}^2 - 2u_{11} u_{13}^2 u_{22}^2 u_{12}^2 + 2u_{11}^2 u_{33}^2 u_{12}^2 + 2u_{22}^2 u_{33}^2 u_{12}^2 - 2u_{11} u_{13}^2 u_{33}^2 u_{12}^2 - 10u_{11} u_{22}^2 u_{33}^2 u_{12}^2 - 38u_{11} u_{23}^2 u_{33}^2 u_{12}^2 - 2u_{22}^2 u_{23}^2 u_{33}^2 u_{12}^2 \\ & - 10u_{11}^2 u_{22} u_{33}^2 u_{12}^2 - 38u_{13}^2 u_{22} u_{33}^2 u_{12}^2 + 72u_{11} u_{13}^2 u_{23}^2 u_{12}^2 - 36u_{13} u_{22} u_{23}^2 u_{12}^2 - 8u_{13} u_{23}^2 u_{33}^2 u_{12}^2 + 12u_{11} u_{13} u_{23} u_{33}^2 u_{12}^2 + 12u_{13} u_{22} u_{23} u_{33}^2 u_{12}^2 \\ & - 36u_{11}^3 u_{23}^2 u_{12}^2 - 8u_{13}^3 u_{22}^2 u_{23} u_{12}^2 + 12u_{11} u_{13}^2 u_{22}^2 u_{23} u_{12}^2 - 8u_{11}^3 u_{13} u_{23} u_{12}^2 + 72u_{13}^3 u_{22} u_{23} u_{12}^2 + 12u_{11}^2 u_{13} u_{22} u_{23} u_{12}^2 - 36u_{13} u_{23}^2 u_{33}^2 u_{12}^2 \\ & - 36u_{13}^3 u_{23} u_{33} u_{12}^2 + 12u_{13} u_{22}^2 u_{23} u_{33} u_{12}^2 + 12u_{11}^2 u_{13} u_{23} u_{33} u_{12}^2 - 48u_{11} u_{13} u_{22} u_{23} u_{33} u_{12}^2 + 4u_{13}^6 + 4u_{23}^6 + u_{11}^2 u_{13}^4 + u_{11}^2 u_{22}^4 + 4u_{13}^2 u_{22}^4 \\ & - 8u_{11}^4 u_{23}^4 + 12u_{13}^2 u_{23}^4 + u_{22}^2 u_{23}^4 + 8u_{11} u_{22} u_{23}^4 + u_{11}^2 u_{33}^4 + u_{22}^2 u_{33}^4 - 2u_{11} u_{22} u_{33}^4 - 2u_{11}^3 u_{33}^3 - 2u_{11}^3 u_{23}^3 - 2u_{22}^3 u_{33}^3 - 2u_{11} u_{13}^2 u_{33}^3 \\ & + 2u_{11} u_{22}^2 u_{33}^3 + 2u_{11} u_{23}^2 u_{33}^3 - 2u_{22} u_{23}^2 u_{33}^3 + 2u_{11}^2 u_{22} u_{33}^3 + 2u_{13}^2 u_{22} u_{33}^3 + u_{11}^4 u_{22}^2 - 8u_{13}^4 u_{22}^2 + 2u_{11}^2 u_{13}^2 u_{22}^2 + 4u_{11}^4 u_{23}^2 + 12u_{13}^4 u_{23}^2 + 2u_{11} u_{13}^2 u_{23}^2 \\ & + 20u_{11}^2 u_{13}^2 u_{23}^2 + 2u_{11}^2 u_{22}^2 u_{23}^2 + 20u_{13}^2 u_{22}^2 u_{23}^2 - 8u_{11}^3 u_{22} u_{23}^2 - 38u_{11} u_{13}^2 u_{22} u_{23}^2 + u_{11}^4 u_{23}^2 + u_{13}^4 u_{23}^2 + u_{22}^2 u_{23}^2 + u_{23}^4 u_{23}^2 + 2u_{11} u_{22}^2 u_{23}^2 + 8u_{11}^2 u_{13}^2 u_{23}^2 \\ & - 6u_{11}^2 u_{22}^2 u_{23}^2 + 2u_{13}^2 u_{22}^2 u_{23}^2 + 2u_{11}^2 u_{23}^2 u_{23}^2 + 2u_{13}^2 u_{23}^2 u_{23}^2 + 8u_{22}^2 u_{23}^2 u_{23}^2 - 10u_{11} u_{22} u_{23}^2 u_{23}^2 + 2u_{11}^3 u_{22} u_{23}^2 - 10u_{11} u_{13}^2 u_{22} u_{23}^2 + 8u_{11} u_{13}^4 u_{22} \\ & + 2u_{11}^3 u_{13}^2 u_{22} - 10u_{11} u_{13}^4 u_{33} - 2u_{11} u_{22}^4 u_{33} + 8u_{11} u_{23}^4 u_{33} - 10u_{22} u_{23}^4 u_{33} + 2u_{11}^3 u_{22} u_{33} - 8u_{13}^2 u_{22} u_{33} - 2u_{11}^3 u_{13} u_{33} + 2u_{11}^3 u_{22} u_{33} \\ & + 20u_{11} u_{13}^2 u_{22} u_{33} - 8u_{11}^3 u_{23} u_{33} - 2u_{22}^3 u_{23} u_{33} - 2u_{11} u_{13}^2 u_{23} u_{33} - 10u_{11} u_{22}^2 u_{23} u_{33} + 20u_{11}^2 u_{22} u_{23} u_{33} - 2u_{13}^2 u_{22} u_{23} u_{33} - 2u_{11}^4 u_{22} u_{33} \\ & + 8u_{13}^4 u_{22} u_{33} - 10u_{11}^2 u_{13}^2 u_{22} u_{33}. \end{aligned}$$

6.3. The case D_•. The next case, in the B–D series, when the ‘exceptionally simple PDE’ \mathcal{E}_Y is not anymore the minimal–degree one, is D₄, which we describe here. To this end, we provide a construction of the minimal–degree equation for all groups of type D. We fix a basis

$$(45) \quad \mathbb{C}^2 = \langle \alpha, \beta \rangle,$$

from which it follows the bi–Lagrangian decomposition

$$(46) \quad \mathbb{C}^2 \otimes \mathbb{C}^n = (\langle \alpha \rangle \otimes \mathbb{C}^n) \oplus (\langle \beta \rangle \otimes \mathbb{C}^n)$$

of the $2n$ –dimensional symplectic space and irreducible $\mathrm{SL}_2 \times \mathrm{SO}_n$ –module $\mathbb{C}^2 \otimes \mathbb{C}^n$. Observe that there are obvious identifications

$$\begin{aligned} \Lambda^n(\mathbb{C}^2 \otimes \mathbb{C}^n) & \equiv \bigoplus_{i=0}^n (\Lambda^i(\langle \alpha \rangle \otimes \mathbb{C}^n) \otimes \Lambda^{n-i}(\langle \beta \rangle \otimes \mathbb{C}^n)) \equiv \bigoplus_{i=0}^n \langle \alpha^i \beta^{n-i} \rangle \otimes \Lambda^i(\mathbb{C}^n) \otimes \Lambda^{n-i}(\mathbb{C}^n) \\ & \equiv \bigoplus_{i=0}^n \langle \alpha^i \beta^{n-i} \rangle \otimes \Lambda^i(\mathbb{C}^n)^{\otimes 2}, \end{aligned}$$

such that the Plücker embedding space $\Lambda_0^n(\mathbb{C}^2 \otimes \mathbb{C}^n)$ of $\mathbb{C}^2 \otimes \mathbb{C}^n$ can be identified with

$$\Lambda_0^n(\mathbb{C}^2 \otimes \mathbb{C}^n) = \bigoplus_{i=0}^n \langle \alpha^i \beta^{n-i} \rangle \otimes S_0^2 \Lambda^i(\mathbb{C}^n).$$

The notation, as well as the decomposition itself, are *different* than the one given in (25). In particular, here $S_0^2 \Lambda^i \mathbb{C}^n$ denotes the kernel of the natural SL_n –invariant map $\Lambda^i \mathbb{C}^n \otimes \Lambda^i \mathbb{C}^n \rightarrow \Lambda^{i-1} \mathbb{C}^n \otimes \Lambda^{i+1} \mathbb{C}^n$ given on decomposable elements by

$$(v_1 \wedge \dots \wedge v_i) \otimes (w_1 \wedge \dots \wedge w_i) \mapsto \sum_{j=1}^i (-1)^{i+j} (v_1 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_i) \otimes (v_j \wedge w_1 \wedge \dots \wedge w_i).$$

It is precisely the SL_n –irreducible summand in $S^2 \Lambda^i \mathbb{C}^n$ whose highest weight is twice the i –fundamental weight (for $i < n$). The Lagrangian n –planes in a favourable position with respect to the splitting (46) are labeled by symmetric $n \times n$ matrices $U = (u_{ij})$. Indeed, if U is understood as a map from $\langle \alpha \rangle \otimes \mathbb{C}^n$ to $\langle \beta \rangle \otimes \mathbb{C}^n$, then its graph is a Lagrangian subspace $L(U)$ nondegenerately projecting over the first space (see Lemma 5 (3)). It is convenient now to introduce the natural extension $U^\bullet : \Lambda^\bullet(\langle \alpha \rangle \otimes \mathbb{C}^n) \rightarrow \Lambda^\bullet(\langle \beta \rangle \otimes \mathbb{C}^n)$ of U to the exterior algebra, and its restrictions $U^{(i)} : \Lambda^i(\langle \alpha \rangle \otimes \mathbb{C}^n) \rightarrow \Lambda^i(\langle \beta \rangle \otimes \mathbb{C}^n)$ to the corresponding i^{th} degree pieces. By Poincaré duality, we also have $U^{(i)} \in \langle \alpha^i \beta^{n-i} \rangle \otimes S^2 \Lambda^i(\mathbb{C}^n)$, and in fact $U^{(i)}$ always lies in the SL_n –irreducible subspace $S_0^2 \Lambda^i(\mathbb{C}^n) \subseteq S^2 \Lambda^i(\mathbb{C}^n)$. Plücker–embedding $L(U)$ into $\mathbb{P}(\Lambda_0^n(\mathbb{C}^2 \otimes \mathbb{C}^n))$ means taking the ‘volume’ of $L(U)$, viz.

$$(47) \quad \det(L(U)) = \left[\sum_{i=0}^n \alpha^i \beta^{n-i} U^{(i)} \right].$$

But now we can use the projection $\pi : \Lambda_0^n(\mathbb{C}^2 \otimes \mathbb{C}^n) \rightarrow S^n \mathbb{C}^2$ to map the representative of (47) into

$$(48) \quad \pi \left(\sum_{i=0}^n \alpha^i \beta^{n-i} U^{(i)} \right) = \sum_{i=0}^n \alpha^i \beta^{n-i} \mathrm{tr} U^{(i)} \in S^n \mathbb{C}^2,$$

where we use the SO_n -invariant quadratic form on \mathbb{C}^n . It remains to observe that $S^n\mathbb{C}^2$ is equipped with a quadratic form, whose matrix in the standard basis $\alpha^n, \alpha^{n-1}\beta, \alpha^{n-2}\beta^2, \dots, \alpha\beta^{n-1}, \beta^n$, is the anti-diagonal one, with entries $c_0, c_1, \dots, c_{n-1}, c_n, c_{n-1}, \dots, c_1, c_0$, where

$$c_k = (-1)^k \binom{n}{k}.$$

Thus, evaluating this form on (48), one gets a quadratic expression

$$F(U) := \sum_{i=0}^n (-1)^i \binom{n}{i} \mathrm{tr} U^{(i)} \mathrm{tr} U^{(n-i)}$$

in the minors of the $n \times n$ symmetric matrix U , i.e., a 2nd order nonlinear PDE of ‘hyperquadric section’ type. For example, for $n = 4$ we have

$$(49) \quad \frac{1}{2}F(U) = \det U - 4 \mathrm{tr}(U) \mathrm{tr}(U^\#) + 3 \mathrm{tr}(U^{(2)})^2,$$

having denoted by $U^\#$ the cofactor matrix of U . Observe that (49) is indeed quadratic, because $\det U$ is multiplied by the ‘minor of order zero’, i.e., by 1 (cf. (19)).

6.4. The case G_2 . In the case of $G = G_2$, the contact grading (4) reads

$$\mathfrak{g} = \mathbb{C} \oplus \underbrace{S^3\mathbb{C}^2}_{\mathfrak{C}_0 = \mathfrak{g}-1} \oplus \underbrace{\mathfrak{sl}(2)}_{\mathfrak{g}_0} \oplus \mathbb{C} \oplus S^3\mathbb{C}^* \oplus \mathbb{C}^*,$$

where the semi-simple part $\mathfrak{sl}(2)$ of $\mathfrak{g}_0 = \mathfrak{gl}(2)$ has been spelt out. Recall that the standard $\mathfrak{sl}(2)$ -module structure on $S^3\mathbb{C}^2$ is precisely the one induced from the bracket with \mathfrak{g}_0 (see Subsection 3.2). The sub-adjoint variety $Y \subset \mathbb{P}\mathbb{C}_0$ coincides with the unique closed $\mathfrak{sl}(2)$ -orbit $\mathbb{P}^1 = \mathbb{P}\mathbb{C}^2 \subset \mathbb{P}S^3\mathbb{C}^2$, which is made of rank-one elements (see Subsection 5.2). In other words, Y is the twisted cubic in \mathbb{P}^3 and hence the minimal-degree 2nd order PDE on the adjoint contact manifold M of G_2 is the Lagrangian Chow transform \mathcal{E}_Y of the field of twisted cubics on M .

In order to write down \mathcal{E}_Y in Darboux coordinates, we choose a bi-Lagrangian decomposition of $S^3\mathbb{C}^2$ and then use Proposition 8. The (conformally) unique symplectic form on $S^3\mathbb{C}^2$ is the one induced by the (conformally) unique symplectic structure on \mathbb{C}^2 , which in turn correspond to the choice of a volume form on \mathbb{C}^2 . By using the same basis (45) as before, we obtain a bi-Lagrangian decomposition

$$S^3\mathbb{C}^2 = \langle \alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3 \rangle = \underbrace{\langle \alpha^3, \alpha^2\beta \rangle}_L \oplus \underbrace{\langle \alpha\beta^2, \beta^3 \rangle}_{L^*} = \langle x^1, x^2 \rangle \oplus \langle -3u_2, u_1 \rangle \cong S^3\mathbb{C}^{2*}.$$

The ideal $\mathcal{I}_Y \subset S^\bullet(S^3\mathbb{C}^{2*})$ is generated by three elements of degree two. Indeed, there is an exact sequence

$$0 \longrightarrow \mathcal{I}_Y \cap S^2(S^3\mathbb{C}^{2*}) \longrightarrow S^2(S^3\mathbb{C}^{2*}) \longrightarrow S^6\mathbb{C}^{2*} \longrightarrow 0$$

of $\mathfrak{sl}(2)$ -modules, decomposing the 10-dimensional $S^2(S^3\mathbb{C}^{2*})$ into irreducible representations. Parametrically, $Y = \{[(t\alpha + s\beta)^3 \mid (t : s) \in \mathbb{P}^1], \text{ that is, } Y = \{[t^3 : t^2s : s^3 : -3ts^2] \mid (t : s) \in \mathbb{P}^1\}$, in the coordinates $[x^1 : x^2 : u_1 : u_2]$. Then it is easy to check that \mathcal{I}_Y is generated by $3x^1u_1 + x^2u_2$, $x^1u_2 + 3(x^2)^2$, $9x^2u_1 - u_2^2$. It remains to apply the Lagrangian Chow-form and discover that

$$\mathcal{E}_Y = \{27u_{11}^2 - u_{12}^2u_{22}^2 + u_{11}u_{22}^3 + 16u_{12}^3 - 18u_{11}u_{12}u_{22} = 0\}.$$

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REFERENCES

- [1] Ilka Agricola. Old and new on the exceptional group G_2 . *Notices Amer. Math. Soc.*, 55(8):922–929, 2008. ISSN 0002-9920.
- [2] D. V. Alekseevskii. Contact homogeneous spaces. *Funktsional. Anal. i Prilozhen.*, 24:74–75, 1990. URL <http://www.ams.org/mathscinet-getitem?mr=1092805>.
- [3] Dmitri V. Alekseevsky, Ricardo Alonso-Blanco, Gianni Manno, and Fabrizio Pugliese. Contact geometry of multidimensional Monge-Ampère equations: characteristics, intermediate integrals and solutions. *Ann. Inst. Fourier (Grenoble)*, 62(2):497–524, 2012. ISSN 0373-0956. doi: 10.5802/aif.2686. URL <http://dx.doi.org/10.5802/aif.2686>.
- [4] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. ISBN 3-540-42650-7. doi: 10.1007/978-3-540-89394-3. URL <http://dx.doi.org/10.1007/978-3-540-89394-3>. Translated from the 1968 French original by Andrew Pressley.

- [5] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. ISBN 3-540-43405-4. Translated from the 1975 and 1982 French originals by Andrew Pressley.
- [6] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior differential systems*, volume 18 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1991. ISBN 0-387-97411-3.
- [7] Jarosław Buczyński. Algebraic Legendrian varieties. *Dissertationes Math. (Rozprawy Mat.)*, 467:86, 2009. ISSN 0012-3862. doi: 10.4064/dm467-0-1. URL <http://dx.doi.org/10.4064/dm467-0-1>.
- [8] Andreas Čap and Jan Slovák. *Parabolic geometries. I*, volume 154 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2009. ISBN 978-0-8218-2681-2. doi: 10.1090/surv/154. URL <http://dx.doi.org/10.1090/surv/154>. Background and general theory.
- [9] Willem A. de Graaf. *Lie algebras: theory and algorithms*, volume 56 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 2000. ISBN 0-444-50116-9. doi: 10.1016/S0924-6509(00)80040-9. URL [http://dx.doi.org/10.1016/S0924-6509\(00\)80040-9](http://dx.doi.org/10.1016/S0924-6509(00)80040-9).
- [10] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. ISBN 0-387-97527-6; 0-387-97495-4. doi: 10.1007/978-1-4612-0979-9. URL <http://dx.doi.org/10.1007/978-1-4612-0979-9>. A first course, Readings in Mathematics.
- [11] Roe Goodman and Nolan R. Wallach. *Symmetry, representations, and invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009. ISBN 978-0-387-79851-6. doi: 10.1007/978-0-387-79852-3. URL <http://dx.doi.org/10.1007/978-0-387-79852-3>.
- [12] Jan Gutt. http://www.cft.edu.pl/p_view/_jgutt/files/Quads.hs, 2016.
- [13] Jan Gutt. http://www.cft.edu.pl/p_view/_jgutt/files/sympluck.lie, 2016.
- [14] Fred W. Helenius. Freudenthal triple systems by root system methods. *J. Algebra*, 357:116–137, 2012. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2012.01.025. URL <http://dx.doi.org/10.1016/j.jalgebra.2012.01.025>.
- [15] Howard Hiller. Combinatorics and intersections of Schubert varieties. *Comment. Math. Helv.*, 57(1):41–59, 1982. ISSN 0010-2571. doi: 10.1007/BF02565846. URL <http://dx.doi.org/10.1007/BF02565846>.
- [16] Giovanni Moreno ([http://mathoverflow.net/users/22606/giovanni moreno](http://mathoverflow.net/users/22606/giovanni%20moreno)). Why there is a relation among the second-order minors of a symmetric 4×4 matrix? MathOverflow, 2015. URL <http://mathoverflow.net/q/209058>. URL: <http://mathoverflow.net/q/209058> (version: 2015-06-12).
- [17] Kazuhiko Koike and Itaru Terada. Young-diagrammatic methods for the representation theory of the classical groups of type B_n , C_n , D_n . *J. Algebra*, 107(2):466–511, 1987. ISSN 0021-8693. doi: 10.1016/0021-8693(87)90099-8. URL [http://dx.doi.org/10.1016/0021-8693\(87\)90099-8](http://dx.doi.org/10.1016/0021-8693(87)90099-8).
- [18] Bertram Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961. ISSN 0003-486X.
- [19] Boris Kruglikov. Finite-dimensionality in Tanaka theory. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(1):75–90, 2011. ISSN 0294-1449. doi: 10.1016/j.anihpc.2010.10.001. URL <http://dx.doi.org/10.1016/j.anihpc.2010.10.001>.
- [20] P.J. Olver. *Classical Invariant Theory*. London Mathematical Society Student Texts. Cambridge University Press, 1999. ISBN 9780521558211. URL <https://books.google.pl/books?id=1G1HYhNRAqEC>.
- [21] Katja Sagerschnig. Split octonions and generic rank two distributions in dimension five. *Arch. Math. (Brno)*, 42 (suppl.):329–339, 2006. ISSN 0044-8753.
- [22] Noboru Tanaka. On differential systems, graded Lie algebras and pseudogroups. *J. Math. Kyoto Univ.*, 10:1–82, 1970. ISSN 0023-608X.
- [23] D. The. Exceptionally simple pde. *ArXiv e-prints*, 2016. URL <http://arxiv.org/abs/1603.08251>.
- [24] Dennis The. Conformal geometry of surfaces in the Lagrangian Grassmannian and second-order PDE. *Proc. Lond. Math. Soc. (3)*, 104(1):79–122, 2012. ISSN 0024-6115. doi: 10.1112/plms/pdr023. URL <http://dx.doi.org/10.1112/plms/pdr023>.
- [25] Burt Totaro. Towards a Schubert calculus for complex reflection groups. *Math. Proc. Cambridge Philos. Soc.*, 134(1):83–93, 2003. ISSN 0305-0041. doi: 10.1017/S0305004102006205. URL <http://dx.doi.org/10.1017/S0305004102006205>.
- [26] Keizo Yamaguchi. Differential systems associated with simple graded Lie algebras. In *Progress in differential geometry*, volume 22 of *Adv. Stud. Pure Math.*, pages 413–494. Math. Soc. Japan, Tokyo, 1993.
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